Approximately j-dimensional Koch type sets are potentially minimal surface singularities.

Amos Koeller

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Chapter 1

Introduction

In the fields of Geometric Measure Theory and Differential Geometry we find that the study of surfaces (Minimal Surfaces, Stationary Surfaces, Energy Minimising Surfaces, etc.) and the flows of surfaces (Mean Curvature flow, Ricci flow, Brakke flow, etc.) play a central role. Such objects are not in general well behaved in that they have initially, or develop in finite time, singularities. Simply speaking, these can be thought of as holes, edges, corners, or in general points that around which no neighbourhood can be described by a graph. It is natural then that an understanding of the structure of such sets would be desired.

In the present state of knowledge surprisingly little is known about these sets. Although, particularly in weak formations, regularity theorems are relatively standard in studies of these objects (See in particular Allard, White, Simon, Brakke, Ecker), this tells us more about how much of the surface we may consider as being smooth (or regular) than about the structure or measure of the singularity set itself.

Some important results on the structure of the singularity sets themselves are due to White, whose stratification results show that the dimension of the singularity set is at least 1 less than that of the surface, and Simon, who has shown that in a particular class of minimal surfaces the singularity set is always a finite union of countably rectifiable sets in the dimension of the singularity.

What is not known is anything at all about the shape of a singularity sets. We do indeed have examples of singularity sets but they are all simple, (i.e. the subset of a line, or a point) which leaves a lot of space between examples and generally provable results.

In his paper showing the rectifiability of singularity sets of a certain class of minimal surfaces, Simon shows that singularity sets can be approximated by planes in the dimension in which they occur. In mean curvature flow, Huisken and Sinestrari have shown that blow ups around singularity points lead to eventually bounding the singularity set (blown up) in a cylinder. This, when considering the axis of the cylinder as a plane in the appropriate dimension is again an approximation to a plane in the dimension of the singularity.

This tells us that the properties of sets that are approximately planes of some dimension are worth considering to see what properties we can get "for free" and what sort of potential problems does one need to be wary of when considering the singularity sets.

As a model for what is meant when we say that a set is approximately a j-dimensional plane or indeed that a set is approximately j-dimensional we use the 'plane like' properties shown by Simon to be possessed by singularity set approximations.

We isolate these properties to construct an ordering of eight strengths of j-dimensional plane approximation of which the property combination specifically used by Simon is the fourth. We classify these definitions in terms of whether or not they ensure actual j-dimensionality and whether or not they ensure locally $\mathcal{H}^j$-finite measure in either a strong or a weak sense.

The definitions allow for the full spectrum of possibilities. The strongest definition implying that the set is locally a finite union of Lipschitz graphs and the weakest two do not even ensure that the set be j-dimensional.

The most interesting case, however, is that of the complications of our fourth definition, intriguingly the same as that arising in Simons work. This definition ensures j-dimensionality, but what makes this case interesting is that while locally finite j-dimensional measure is not ensured, any counter examples are necessarily exotic. We show that while satisfying 'approximately j-dimensional' properties such sets have points of infinite $\mathcal{H}^j$-density but that no piece of any Lipschitz graph may pass through such a point. This rules out any vaguely well behaved sets (or countable unions of vaguely well behaved sets) from both satisfying our fourth definition and failing to have locally finite $\mathcal{H}^j$-measure.

Since our classification is complete it follows that we can (and indeed do) provide a set satisfying this fourth definition that also does not have locally finite j-dimensional measure. The set is a variation on the fractal known as the Koch set. Since all singularity sets are closed we go on to show that a closed version of this counter example exists which implies that in principle singularity sets could be as terribly behaved as the counter example.

Especially since, at least in the minimal surface case, singularity sets are known to be finite unions of countably j-rectifiable sets (see [16]) the question of whether such sets as these counter examples are finite unions of countably j-rectifiable sets (and so continue to, potentially, be singularity sets) becomes of interest.

The answer to this question for the particular examples initially given turns out to be no, they are not rectifiable without considering measure conditions and so cannot be finite unions of countably j-rectifiable sets. However, since the explicitly constructed counter examples are members of a family of constructions this by no means rules out the possibility of very poorly behaved singularity sets.

The second part of the work then defines generalisations of the construction of the constructed counter examples. We call these sets, due to their similarity to the Koch sets, Koch-type sets. We then concentrate on giving dimension, measure and rectifiability conditions for these generalised sets.

We find, encouragingly for the study of singularity sets that should such a set be first of all rectifiable then it can also be written as a single Lipschitz graph.

This would immediately imply, since we need to remove the 'corners' of the sets in order to satisfy our fourth definition that any singularity set that may be of a Koch type set form should also be a subset of a single Lipschitz graph.

The structure of the work is as follows:

In chapter 2 we present a more precise formulation of the motivating mathematics including some particularly relevant standard general geometric measure theoretic definitions and results, provide the list of definitions as well as the results already known in terms of our classification aims and results from which looked for classification results are a short corollary.

In chapter 3 we construct the specific counter examples that will be used

in our classifications including the explicit examples of Koch type sets mentioned above. We go on to prove some important properties of these sets. Some properties, for example dimension, follows from some relatively general previous results of Hutchinson (see [10]). Since it is often instructive to see the direct proof for explicit examples we provide direct proofs for these results as well.

Before moving on to show that the counter examples do indeed satisfy the definitions that they are counter examples to, a by no means trivial task, we show in chapter 4 that the complexity of the counter examples constructed is indeed necessary; in that no 'simple' example could possibly suffice. Further, we show the path to showing that singularity sets have locally finite measure is shorter than was previously thought, in that we need only show that the set is graph possesing at all points of infinite density. This is shorter than previously thought since such a property is so very weak. It does not even require that the set be weakly locally countably rectifiable.

In chapter 5 we fit the counter examples to their respective definitions and complete the task of classifying the definitions.

Chapter 6 gathers a few other miscellaneous relevant results and describes dimension generalisation of the explicit counter examples which are constructed to satisfy approximations to dimension 1 (though, of course, some are actually of fractal dimension between 1 and 2.)

Finally, in Chapters 7 and 8, we deal with the question of dimension, measure and rectifiability of the family of sets that are the generalised form of the explicit counter examples given. These generalisations are divided into two levels of generalisation, first and second degree variation. We keep the two levels of generalisation distinct since, although first degree variation generalisations are also second degree variation generalisations, they allow for stronger results. This is because much more can 'go wrong' in the second degree variation case.

Chapter 2

Background, Definition and Existing Results

2.1 Preliminary Geometric Measure Theory

We start straight of with some relevant measure theoretic background. The standard references are of course [15] and [7]. We assume basic familiarity with general measure theory and we use the usual symbol for r-dimensional Hausdorff measure $\mathcal{H}^r$ for $r \in \mathbb{R}$. Also, we denote the Hausdorff volume of the unit n-ball by ω_n.

As mentioned, a major part of our investigation regards dimension, for which we are interested in Hausdorff dimension which we define as follows.

Definition 2.1.
Set $A \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. Then the **Hausdorff dimension** *of A is defined as*

$$\begin{aligned} dimA &:= \inf\{r \in \mathbb{R} : \mathcal{H}^r(A) = 0\} \\ &= \sup\{r \in \mathbb{R} : \mathcal{H}^r(A) = +\infty\} \end{aligned}$$

Another important quantity that we will be using is density, and indeed n-dimensional density.

Definition 2.2.
Let $(X, \mathcal{B}, \mu)$ be a measure space.Then for any subset A of X, and any point $x \in X$, we define the n-dimensional upper and lower n-dimensional densities $\Theta^{*n}(\mu, A, x)$, $\Theta^{n}_{*}(\mu, A, x)$ respectively by

$$\Theta^{*n}(\mu, A, x) = \limsup_{\rho \to 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}$$

and
$$\Theta^n_*(\mu, A, x) = \liminf_{\rho \to 0} \frac{\mu(A \cap B_\rho(x))}{\omega_n \rho^n}.$$
In the case that the two quantities are equal we call the common quantity the n-dimensional μ-density of A at x denoted by $\Theta^n(\mu, A, x)$.

Remark:
Depending on which quantities are from the context understood, we will also use the terms density of A at x or simply the density at x.
The σ-algebra $\mathcal{B}$here is mentioned for formality but is unimportant in the definition.

Also fundamental to our considerations is the concept of rectifiability. We will need several forms of the definition of rectifiability. Their equivalences are well presented in [15]. We shall not here be interested in general rectifiable sets, so we restrict ourselves immediately to countably rectifiable sets. Firstly and most basically we have the following definition.

Definition 2.3.
A set $M \subset \mathbb{R}^{n+k}$ is said to be countably n-rectifiable if
$$M \subset M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n)$$
where $F_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ are Lipschitz functions and $\mathcal{H}^n(M_0) = 0$

Remark By standard Lipschitz extension results we know that we can also write
$$M = M_0 \cup \bigcup_{j=1}^{\infty} F_j(A_j)$$
for subsets $A_j \subset \mathbb{R}^n$.

Notice that we have not required that the sets by measurable, which is occasionally required in definitions of rectifiable sets. It is however not necessary since, as we will see, all of the relevant sets we will be considering are in any case measurable since they can be shown to be expressable as countable unions and intersections of Borel sets in the appropriate Euclidean space.

From this basic definition it is known that the following expression for rectifiable sets holds.

Lemma 2.1.
M is countably n-rectifiable if and only if

$$M \subset \bigcup_{j=0}^{\infty} N_j,$$

where $\mathcal{H}^N(N_0) = 0$ and where each N_j, $j \geq 1$, is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+k}$.

To introduce the final representation that we need we first need the following definitions.

Definition 2.4.
We let the time fixed blow-up function be denoted by η, that is for any subeset $A \subset \mathbb{R}^n$

$$\eta_{y,\rho}(A) = \rho^{-1}(A - y).$$

Let L be an subspace of $\mathbb{R}^n$ and $\rho \in \mathbb{R}$, $\rho > 0$, then

$$L^\rho = \{x \in \mathbb{R}^n : |x - y| < \rho \text{ for some } y \in L\}.$$

and

Definition 2.5.
If M is an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+k}$ and θ is a positive locally $\mathcal{H}^n$-integrable function on M, then we say that a given n-dimensional subspace P of $\mathbb{R}^{n+k}$ is the approximate tangent space for M with respect to θ if

$$\begin{aligned}
\lim_{\lambda \to 0} \int_{\eta_{x,\lambda} M} f(y)\theta(x + \lambda y) d\mathcal{H}^n(y) &:= \lim_{\lambda \to 0} \lambda^{-n} \int_M f(\lambda^{-1}(z - x))\theta(z) d\mathcal{H}^n(z) \\
&= \theta(x) \int_P f(y) d\mathcal{H}^n(y)
\end{aligned}$$

for all $f \in C^0_C(\mathbb{R}^{n+k})$. The function θ is called the multiplicity function of M.

We will in general consider sets with the multiplicity function set to 1.

Our final definition of countably n-rectifiable sets is now stated in the form of the following theorem.

Theorem 2.1.
Suppose M is $\mathcal{H}^n$-measurable. Then M is countably n-rectifiable if and only if there is a positive locally $\mathcal{H}^n$-integrable function θ on M with respect to which the approximate tangent space $T_X M$ exists for $\mathcal{H}^n$-a.e. $x \in M$.

Remark: We note that, for example in [16] it is often required that the total or $\mathcal{H}^n$ measure of a set M or at least $\mathcal{H}^n(M \cap K)$ be finite for each compact set K. We do not, a priori make this assumption.

Rectifiability can be seen as the weakest form of structure that a set can posses. However, we can explore parts of even unrectifiable sets in the case that they contain rectifiable parts. This fact will be useful to us. Particularly in chapter 4. For these reason we also define purely unrectifiable sets.

Definition 2.6.
A set P is said to be purely n-unrectifiable if it contains no countably n-rectifiable subsets of $\mathcal{H}^n$ positive measure.

We note to this definition that for any set in $\mathbb{R}^{n+k}$, A, A can always be decomposed into the disjoint union of two sets $A = R \cup P$ where R is countably R rectifiable and P is purely n-unrectifiable.

2.2 Motivation of the Classification

We move now onto the motivation and construction of the problem at hand, previous results and results that follow more or less trivially from the literature.

An additional motivation to that mentioned in the introduction to this work was to perhaps uncover a way to attack the local $\mathcal{H}^j$-finality of singularity sets for minimal surfaces or surfaces moving by their mean curvature. This is supported by the mentioned results in Leon Simons' [16] paper on the rectifiability of minimal surfaces, and recent work by Huisken and Sinistrari that shows that estimates on the shape of singularity sets is heading in the direction of satisfying the properties of the definitions under consideration. In particular, in Simon [16] a Lemma (the same one as has been previously discussed) shows that at least parts of the singularity sets of particular types of minimal surfaces exactly satisfy one of the approximation properties.

We state this Lemma (after appropriate definitions) as a motivational starting point and also as it highlights some of the interesting points of the results. We then state the definitions mentioned in the introduction that we wish to classify and provide more fully a discussion of what it is we want to classify in these definitions. We also provide here a summary of the classification that is the central classification of the work.

Definition 2.7.
By a **Multiplicity one class** *of minimal surfaces,* $\mathcal{M}$*, we will mean a set of smooth (i.e. infinitely differentiable) n-dimensional minimal submanifolds. Each* $M \in \mathcal{M}$ *is assumed to be properly embedded in* $\mathbb{R}^{n+k}$ *in the sense that for each* $x \in M$ *there is a* $\sigma > 0$ *such that* $M \cap \overline{B}_\sigma(x)$ *is a compact connected embedded smooth manifold with boundary contained in* $\partial B_\sigma(x)$*. We also assume that for each* $M \in \mathcal{M}$ *there is a corresponding open set* $U_M \supset M$ *such that* $\mathcal{H}^n(M \cap K) < \infty$ *for each compact* $K \subset U_M$*, and such that* M *is stationary in* U_M *in the sense that*

$$\int_M div_M \Phi d\mu = 0.$$

whenever $\Phi = (\Phi^1, ..., \Phi^{n+k}) : U_M \to \mathbb{R}^{n+k}$ *is a* C^∞ *vector field with compact support in* U_m*. Where we have used* $\mu = \mathcal{H}^n|_M$*. We also require that the multiplicity one class of submanifolds are closed with respect to sequential compactness, orthogonal transformations and homotheties, that is:*

1. $M \in \mathcal{M} \Rightarrow q \circ \eta_{x,\rho} M \in \mathcal{M}$ *and* $q \circ \eta_{x,\rho} U_M = U_{q \circ \eta_{x,\rho} M}$ *for each* $\rho \in (0, 1]$*, and for each orthogonal transformation* q *of* $\mathbb{R}^{n+k}$*.*
2. *If* $\{M_j\}_j \subset \mathcal{M}$*,* $U \subset \mathbb{R}^{n+k}$ *with* $U \subset U_M$*, for all sufficiently large* j*, and* $\sup_{j \geq 1} \mathcal{H}^n(M_j \cap K) < \infty$ *for each compact* $K \subset U$*, then there is a subsequence* $M_{j'}$ *and an* $M \in \mathcal{M}$ *such that* $U_M \supset U$ *and* $M_{j'} \to M$ *in* U *in the sense that*

$$\int_{M_{j'}} f d\mathcal{H}^n \to \int_M f d\mathcal{H}^n$$

for any $f \in C^0_C(U, \mathbb{R})$*.*

We assume here that the $M \in \mathcal{M}$ have no removable singularities: thus if $x \in \overline{M} \cap U_M$ and there is a $\sigma > 0$ such that $\overline{M} \cap \overline{B}_\sigma(z)$ is a smooth connected embedded n -dimensional submanifold with boundry contained in $\partial B_\sigma(z)$, then $z \in M$. Subject to this agreement we can make the following definition

Definition 2.8.
Suppose that $\mathcal{M}$ *is as above and that* $M \in \mathcal{M}$ *then the* **(interior) singular set of** M *(relative to* U_M*) is defined by*

$$singM = U_M \cap \overline{M} \sim M$$

and the regular set of M *is simply* M *itself, that is*

$$regM = M.$$

With these definitions we can now state the motivating Lemma due to Simon [16]:

Lemma 2.2.
If $\mathcal{M}$ is a multiplicity one class of minimal surfaces, $M \in \mathcal{M}$,

$$m := \max\{dimsingM : M \in \mathcal{M}\}$$

$$z_0 \in singM$$

and

$$S_+(z_0) := \{z \in M : \Theta^m(M,z) \geq \Theta^m(M,z_0)\}$$

Then for each $\varepsilon > 0$ there is a $p = p(\varepsilon, z_0, M) > 0$ such that $S_+(z_0)$ has the following approximation property in $\overline{B_p(z_0)}$:

For each $\sigma \in (0,p]$ and $z \in S_+(z_0) \cap \overline{B_p(z_0)}$ there is an m-dimensional affine subspace $L_{z,\sigma}$ containing z with

$$S_+ \cap B_\sigma(z) \subset \text{ the } (\varepsilon\sigma) - nhood\ of\ L_{z,\sigma}.$$

We note that in the case of Mean Curvature Flows, the singularity set can also be defined as follows:

Definition 2.9.
We say that a solution of Mean Curvature Flow $(M_t)_{t<t_0}$ reaches $x_0 \in \mathbb{R}^{n+1}$ at time t_0 if there exists a sequence (x_j, t_j) with $t_j \nearrow t_0$ so that $x_j \in M_j$ and $x_j \to x_0$.

Definition 2.10.
Let $\mathcal{M} = (M_t)$ be a smooth solution of mean curvature flow in $U \times (t_1, t_0)$. We say that $x_0 \in U$ is a **singular point** *of the solution at time t_0 if $\mathcal{M}$ reaches x_0 at time t_0 and has no smooth extension beyond time t_0 in any neighbourhood of x_0. All other points are called* **regular points**. *The* **singular set** *at time t_0 will be denoted by $sing_{t_0}\mathcal{M}$ and the regular set by $reg_{t_0}\mathcal{M}$.*

As singularity sets are the motivation rather than the subject of our investigation, the properties of singular sets are used very little. However, in determining how applicable our results may be to singular sets we find that it is important to note that singular sets (from either definintion) are closed.

Proposition 2.1.
Singular sets as defined in either Definition 2.8 or Definition 2.10 are closed.

Proof:
Suppose that the statement is not true, then there is a point $x \in regM$ such that for all $r > 0$ $B_r(x) \cap singM \neq \emptyset$. In particular since $x \in regM$ there is a radius $\rho_x > 0$ such that $\bar{M} \cap \overline{B_{\rho_x}(x)}$ is "smooth" (either in the infinitely differentiable in space time sense for mean curvature flow, or the sense outlined in Definition 2.7, depending on whether we are proving the result for Definition 2.8 or 2.10) and such that $B_{\rho_x} \cap singM \neq 0$. Thus there is a $z \in singM$ and $\rho_z > 0$ such that $B_{\rho_z}(z) \subset B_{\rho_x}(x)$. It follows that $\bar{M} \cap B_{rho_z}(z)$ is "smooth" and thus $z \in regM$. This contradiction shows such a point x cannot be found which completes the proof. $\diamondsuit$

We now construct the properties that we will be investigating. We will always be considering sets being approximated by j-dimensional affine spaces that are subspaces of $\mathbb{R}^n$. We will identify $\mathbb{R} \times \{0\}$ with $\mathbb{R}$ and denote the projection onto $\mathbb{R}$ by π_x. Further, if L is a 1-dimensional affine space in $\mathbb{R}^2$ we will denote the projection onto L by π_L.

Definition A.
Let $A \subset \mathbb{R}^n$ be an arbitrary set and $\delta > 0$; then

(i) A has the weak j-dimensional δ-approximation property if for all $y \in A$ there is $\rho_y > 0$ such that for all $\rho \in (0, \rho_y]$, $B_\rho(y) \cap A \subset$ the $\delta\rho$-neighbourhood of some j-dimensional affine space $L_{y,\rho}$ containing y.

(ii) A has the weak j-dimensional δ-approximation property with local ρ_y-uniformity if for all $y \in A$ there is a $\rho_y > 0$ such that for all $\rho \in (0, \rho_y]$ and all $x \in B_{\rho_y}(y)$

$$B_{\rho_y}(x) \cap A \subset L_{x,\rho}^{\delta\rho}$$

for some j-dimensional affine space $L_{x,\rho}$.

(iii) A is said to have the fine weak j-dimensional δ-approximation property if for all $\delta > 0$ A has the weak j-dimensional δ-approximation property with respect to δ.

(iv) A has the fine weak j-dimensional approximation property with local ρ_y-uniformity if A satisfies (ii) for all $\delta > 0$.

(v) The property (i) is said to be ρ_0-uniform, if A is contained in some ball of radius ρ_0 and if, for every $y \in A$ and every $\rho \in (0, \rho_0]$, $B_\rho(y) \cap A \subset$ the $\delta\rho$-nhood of some j-dimensional affine space $L_{y,\rho}$ containing y.

(vi) A has the strong j-dimensional δ-approximation property if for each $y \in A$ there is a j-dimensional affine space L_y containing y such that the definition (i) holds with $L_{y,\rho} = L_y$ for every $\rho \in (0, \rho_y]$.

(vii) A has the strong j-dimensional δ-approximation property with local ρ_y-uniformity if for all $y \in A$ there exists a $\rho_y > 0$ and an affine space L_y such that for all $x \in B_{\rho_y}(y)$ and all $\rho \in (0, \rho_y]$

$$B_\rho(x) \cap A \subset L_y^{\delta\rho}.$$

(viii) The property in (vi) is said to be ρ_0-uniform if A is contained in some ball of radius ρ_0 and if for each $y \in A$ there is a j-dimensional affine space L_y containing y such that $B_\rho(y) \cap A \subset$ the $\delta\rho$-nhood of L_y for each $\rho \in (0, \rho_0]$.

Due to the long names of the properties, they will be henceforth referred to only by their number.

Our classification is to gat a simple yes or no answer for each of the eight definitions with respect to two questions.

Question 2.1.
We wish to classify the definitions in Definition 1 with respect to the following questions:

1. *if the set will be of dimension j (or rather $\leq j$), and*
2. *if the set will have some locally finite Hausdorff measure property.*

With these questions in mind we will concern ourselves with asking about the answer to (1) or (2) with respect to a certain definition, for example the answer to (i) (1) is no.

As we are generally probing here for 'free information' about singularity sets, and the use of more than one definition of the terms about which we are asking in the literature we remain open as to which definition it is that we are making classifications with respect to. We therefore allow for two strengths of locally finite $\mathcal{H}^j$ measure. In only one case do find that the answer as to possesing locally finite $\mathcal{H}^j$ measure is affected by the choice of strength of definition, that is for (vii) where the definition ensures satisfaction of the weaker but not the stronger. The definitions are:

Definition 2.11.
A subset $A \subset \mathbb{R}^n$ is said to have **locally finite $\mathcal{H}^j$ measure** *(or local $\mathcal{H}^j$-finality) if for all compact subsets $K \subset \mathbb{R}^n$,*

$$\mathcal{H}^j(K \cap A) < \infty,$$

or equivalently, if for all $y \in \mathbb{R}^n$ there exists a radius $\rho_y > 0$ such that

$$\mathcal{H}^j(B_{\rho_y}(y) \cap A) < \infty.$$

A subset is said to have **weak locally finite $\mathcal{H}^j$ measure** *(or weak local $\mathcal{H}^j$-finality) if for each $y \in A$ there exists a radius $\rho_y > 0$ such that*

$$\mathcal{H}^j(B_{\rho_y}(y) \cap A) < \infty.$$

An example of the difference is that

$$\mathcal{N} := \bigcup_{n=1}^{\infty} \mathbb{R} \times \left\{\frac{1}{n}\right\}$$

has weak local $\mathcal{H}^j$-finality but not local $\mathcal{H}^j$-finality. The use of allowing the weak definition is that in some cases, such as the one just given a set with weak local $\mathcal{H}^j$-finality will be the finite union of a collection of sets with local $\mathcal{H}^j$-finality. Which still counld be understood as having reasonably behaved local measure when the structure giving the locally infinite measure is known.

As we will see, and has been hinted at, we do not necessarily get very much information for free. Paritulcarly as we get a "no" to answer the definition corresponding Simons' Lemma. However, as mentioned in the introduction. In this case we do show that in order for something to go wrong the set does have to be truly badly behaved which should be helpful. We now note formally that the condition in Simons' Lemma is definition (iv).

Proposition 2.2.
The $S_+(z_0)$ sets introduced in Lemma 2.2 are (iv).

Proof:
Direct comparison between the property shown in Lemma 2.2 and (iv) shows that this is exactly what is shown in Lemma 2.2. $\diamondsuit$

2.3 Results Following from the Literature

Although the problem we are looking at has not previously been systematically investigated, a few of the results follow easily from results already in the literature for which proofs can be found, for example in Simon [17]. Excepting a counter example, the relevant results can be convieniently stated in the following Lemma.

Lemma 2.3.
(i) There is a function $\beta : [0,\infty) \to [0,\infty)$ *with* $\lim_{\delta \searrow 0} \beta(\delta) = 0$ *such that if* $A \subset \mathbb{R}^n$ *has the* j*-dimensional weak* δ*-approximation property for some given* $\delta \in (0,1]$*, then* $\mathcal{H}^{j+\beta(\delta)}(A) = 0$*. (In particular if* A *has the* j*-dimensional weak* δ*- approximation property for each* $\delta > 0$*, then dim*$A \le j$*.)*

(ii)If $A \subset \mathbb{R}^n$ *has the strong* j*-dimensional* δ*-approximation property for some* $\delta \in (0,1]$*, then* $A \subset \cup_{k=1}^{\infty} G_k$*, where each* G_k *is the graph of some Lipschitz function over some* j*-dimensional subspace of* $\mathbb{R}^n$*.*

(iii) If $A \subset \mathbb{R}^n$ *has the* ρ_0 *uniform strong* j*-dimensional* δ*-approximation property for some* $\delta \in (0,1]$*, then* $A \subset \cup_{k=1}^{Q} G_k$*, where* G_k *is the graph of some Lipschitz function over some* j*-dimensional subspace of* $\mathbb{R}^n$*, L.*

We show in the following Corollary that the above Lemma allows us to answer yes to properties (vi) (1), (viii) (1) and (2), (iii) (1), (iv) (1) and (vii) (1) and (2), although we answer yes to (vii) (2) only with weak local $\mathcal{H}^j$-finality, to local $\mathcal{H}^j$-finality we answer no.

Corollary 2.1.
The answer to the following Definitions is yes:

(vi) (1),

(viii) (1),

(viii) (2),

(iii) (1),

(iv) (1),

(vii) (1), and

(vii) (2)

Proof:
(iii) (1) follows from Lemma 2.3 (i) isince

"*In particular if* A *has the* j*-dimensional weak* δ*- approximation property for each* $\delta > 0$*, then dim*$A \le j$*.*"

means that should A satisfy (iii), then dim$A \le j$ which proves that the answer to (iii) (1) is yes. Further, since (iv) (1) is a strengthening of (iii),

sets satisfying the properties of (iv) must further satisfy any properties following from sets satisfying (iii), thus the answer to (iv) (1) must also be yes.

Any graph of a Lipschitz function over a j-dimensional affine space clearly has dimension less than or equal to j. It follows then that any countable union of such graphs will also have dimension bounded above by j. It thus follows from Lemma 2.3 (ii) and (iii) that the answers to (vi) (1) and (viii) (1) are yes. Similarly to the preceeding paragraph, the fact that (vii) is a strengthening of (vi) that the answer to (vii) (1) is yes.

Further concerning (viii), suppose that we have a set A satifying the conditions of property $(viii)$. Suppose also that $x \in \mathbb{R}^n$ and $\rho > 0$. Then we know that

$$A \cap B_\rho(x) \subset \bigcup_{k=1}^{Q} g_k(\pi_{L_k}(B_\rho(x)))$$

where L_k are the j dimensional affine spaces that Lemma 2.3 ensures exist and the g_k are the Lipschitz functions over the L_k that combined contain A. Thus

$$\mathcal{H}^j(A \cap B_\rho(x)) \leq \sum_{k=1}^{Q} \mathcal{H}^j(g_k(\pi_{L_k}(B_\rho(x)))).$$

Since $card(\{g_k\}_{k=1}^{Q}) = Q < \infty$ there exists a

$$M = \max_k \ \mathrm{Lip} g_k < \infty$$

so that by the Area formula

$$\begin{aligned}
\mathcal{H}^j(A \cap B_\rho(x)) &\leq \sum_{k=1}^{Q} \mathcal{H}^j(g_k(\pi_{L_k}(B_\rho(x)))) \\
&\leq \sum_{k=1}^{Q} M\mathcal{H}^j(\pi_{L_k}(B_\rho(x))) \\
&= \sum_{k=1}^{Q} M\omega_j\rho^j \\
&= QM\omega_j\rho_j \\
&< \infty.
\end{aligned}$$

We thus have that property $(viii)$ does ensure locally finite measure, and thus we have shown that the answer to (viii) (2) is yes.

Finally we note that should we have a set satisfying (vii), then, by definition, for each $y \in A$ there is a $\rho_y > 0$ and an affine space L_y such that for all $x \in B_{\rho_y}(y)$ and all $\rho \in (0, \rho_y]$

$$B_\rho(x) \cap A \subset L_y^{\delta\rho}.$$

It follow that $B_{\rho_y}(y) \cap A$ satisfies (viii), thus $\mathcal{H}^j(K \cap A) < \infty$ for each compact $K \subset \mathbb{R}^n$, That is

$$\begin{aligned} \mathcal{H}^j(B_{\rho_y}(y) \cap A) &\leq \mathcal{H}^j(\overline{B}_{\rho_y}(y) \cap A) \\ &< \infty. \end{aligned}$$

Thus giving weak local $\mathcal{H}^j$-finality, and thus allowing us to answer (vii) (2) with yes. $\diamondsuit$

Remark: We note that the proof as written is also optimal in that we cannot get better than weak local $\mathcal{H}^j$-finality for (vii) as seen in the already given example of $\mathcal{N}$. For each $y \in \mathcal{N}$ we can find a $\rho_y > 0$ such that $B_{\rho_y}(y) \cap \mathcal{N} \subset \mathbb{R} \times \{1/n\}$ for some $n \in \mathbb{N}$, and by setting L_y as this affine space for each y it is clear that $\mathcal{N}$ satisfies (vii), However, for each $r > 0$

$$\mathcal{H}^j(B_r((0,0)) \cap \mathcal{N}) = \infty$$

so that $\mathcal{N}$ does not have locally finite $\mathcal{H}^j$-measure.

Another contribution that comes from Simon [17] is a set that is similar in form to the main and most interesting counter example that is presented here. Its actual construction and properties will be discussed in the following section, however, in noting results that have already been essentially shown, we acknowledge its existence and that it was known to satisfy one of the definitions.

Lemma 2.4.
There is a set, Γ_ε that satisfies (i) for $j = 1$ that has dimension greater than 1.

In later chapters dimension of Γ_ε and related sets will be discussed. The original proofs that we present will be based on the knowledge of how to calculate the dimension of Γ_ε. The proof of the relevant formula will, however, not be presented, as it also already exists in the literature. The proof can be found in [10].

Corollary 2.2.
The answer to (i) (1) and (i) (2) is no.

Proof:
The set Γ_ε of Lemma 3 constructed in the following section provides a counter example to the answer to (i) (1) being yes. Since the dimension of a set satisfying (i) with $j = 1$ could be greater than 1, there is clearly no gaurantee of any form of finite $\mathcal{H}^1 = \mathcal{H}^j$ measure. Thus the answer to (i) (2) is also no. $\diamondsuit$

This completes the survey of the results that were already known, or rather, at least already almost known. So that the complete classification of all the definitions is presented in a convenient easily digested way somewhere we complete this chapter with a table of the complete classification that we prove in this thesis.

The classification of the definitions in Definition 1 with respect to the questions presented in Questions 1 is as follows:

$$\begin{array}{rl}
(i) & (1)\ \ no\ \ (2)\ \ no \\
(ii) & (1)\ \ no\ \ (2)\ \ no \\
(iii) & (1)\ \ yes\ \ (2)\ \ no \\
(iv) & (1)\ \ yes\ \ (2)\ \ no \\
(v) & (1)\ \ no\ \ (2)\ \ no \\
(vi) & (1)\ \ yes\ \ (2)\ \ no \\
(vii) & (1)\ \ yes\ \ (2)\ \ yes(weak)/no(strong) \\
(viii) & (1)\ \ yes\ \ (2)\ \ yes.
\end{array}$$

We note that those definitions classified as yes have all already been answered. It remains only to show that the classification of the remaining definitions is no.

Chapter 3

Construction of the Counter Examples

Having answered all the questions that will be answered with yes, we now turn our attention to providing counter examples for the remaining questions so as to answer no to each of these. For those with a didactic turn of mind, of course these counter examples were constucted in association with answering our questions and not constructed before hand, only to be quite coincidently successfully used later.

The sets being considered are not all trivial sets to construct or to understand. At least not at first sight. We therefore provide only the constructions and some intrinsic properties of the sets, leaving the proofs that they actually satisfy the definitions that they are respectively intended to be counterexamples to until later. For the more complicated sets, particularly A_ε, there is more than one method to construct the set. Some of these will be discussed further in Chapters 7 and 8. For now, however, we satisfy ourselves with the definitions most easily used to fit the constructed sets to the relevant definitions and thus complete the classification.

In this chapter we construct 3 sets and 3 1-parameter families of sets. Of the latter three the first is our own construction of a known set, the same that appears in Lemma 3, which we provide since the necessary properties for our purposes are more easily proven with our construction method. The latter two are then variations of the same set allowing for important extra properties by adding another point of variation. For the sets with a variable there is a range of values of the parameter (independent of which set) for which each resultant specific example is appropriate for our purposes. We will, however, calculate with the parameter left arbitrary since it provides

more generality and makes no difference to the proofs of the results that we want to prove with these sets.

The three simpler sets are of little interest apart from the fact that they are appropriate counter examples to particular definitions. The other three are of independent interest. As well as allowing us to show that some good behaviour is ensured by the approximate j-dimensionality of the sets if not as definite as we had hoped, they provide a range of interesting results on dimension, rectifiability and measure density. Alot of the general proofs concerning properties of these sets are included in the discussion of generalised Koch Type sets (the generalisations of these three 'specific' examples) in Chapters 7 and 8. We include in any case the direct proofs of the properties that we are interested that are relevant to the classification work. That is we include direct proofs that for each definition for which there is a counter example there is a closed counter example (important, since as we have shown in Proposition 2.1 singularity sets are all closed) and that the sets of integer dimension are shown directly to have their respective dimensions.

We construct firstly the three simpler sets. We then construct Γ_ε which will be a counter example to (i) (1) followed by a property of Γ_ε important to our study. We then construct the second more complicated set A_ε which is a counter example to (iv) (2). Since A_ε is not closed and is therefore not possibly a singularity set we make the third construction $\mathcal{A}_\varepsilon$, which is a subset of the second, constructed to be closed but retain the necessary properties. We then prove some necessary proprties of A_ε and $\mathcal{A}_\varepsilon$.

3.1 Simple and Known Sets

The first set has already been defined, and is:

$$\mathcal{N} := \bigcup_{n=1}^{\infty} \mathbb{R} \times \left\{ \frac{1}{n} \right\}$$

Note that we will henceforth identify $R^n \times [0]^{N-n}$ with $\mathbb{R}^n$ in $\mathbb{R}^N$ for each choice of $n, N \in \mathbb{N}$ with $n < N$. The other simple ones, are used in a similar way to $\mathcal{N}$ but need differing levels of fineness approximation with bad properties at one point. Being a collection of flat sheets, $\mathcal{N}$ does not have this property, we therefore define the subset of $\mathbb{R}^2$ defined for each $\delta > 0$ as

$$\Lambda_\delta = \bigcup_{n=1}^{\infty} \text{graph} \left\{ \frac{\delta x}{n} \right\} \cup \bigcup_{n=1}^{\infty} \text{graph} \left\{ \frac{-\delta x}{n} \right\}$$

and the subset of $\mathbb{R}^2$ defined as

$$\Lambda^2 = \bigcup_{n=1}^{\infty} \text{graph}\left\{\frac{x^2}{n}\right\} \cup \bigcup_{n=1}^{\infty} \text{graph}\left\{\frac{-x^2}{n}\right\}.$$

We now construct the more complicated examples, they are both based on the "Koch Curve" which was originally constructed as a fractal set being of dimension between 1 and 2. The first we construct is the set Γ given by Simon in [17], on which the remaining sets are based. The second set, which is actually a function from $\mathbb{R}^+$ into $2^{\mathbb{R}^n}$ (that is, the set is constructed with respect to a variable $\varepsilon \in \mathbb{R}^+$) will be denoted A_ε, and is used as a counter example to (iv) (2). Although Γ_ε was actually constructed as a fixed set, we will allow the set to be constructed with respect to a variable ε, which will later allow us to find appropriate counter examples with respect to (i) (1) for any given δ. The variable set will then be denoted Γ_ε

These constructions rely heavily on the use of triangles so we first make the following definition.

Definition 3.1.
*Let $L = (a, b) = ((a_1, a_2), (b_1, b_2))$ be a line in $\mathbb{R}^2$. A ε-**triangular cap** or, when the context is clear, simply a **cap** will be the triangle , T, with vertices a, b and $c+(a+b)/2$ (we write c also as (c_1, c_2)), where c is chosen such that*

$|c| = \varepsilon$ *and*

$< c, b - a >= 0.$

Further to ensure that the cap is well defined we choose c from the two remaining possible points in $\mathbb{R}^2$ as follows. Should L be an edge of a previously constructed triangular cap, T_0, then c is chosen such that $T \subset T_0$. Otherwise, if $c_1 \not\equiv (a_1+b_1)/2$ (regardless of which of the two possbilities) then c is chosen such that $c_1 > (a_1 + b_1)/2$, otherwise we choose c such that $c_2 > (a_2 + b_2)/2$.

Construction 3.1. .
We construct the set Γ_ε as follows.

Let $\varepsilon > 0$. We begin with a ε-triangular cap, T_0, constructed over the line $A_{0,1} := ((0, 0), (1, 0))$. We then name the two new edges $A_{1,j}$, $j = 1, 2$. We denote the first "approximation", which is T_0, as A_0. We note that $l := \mathcal{H}^1(A_{1,j}) < \mathcal{H}^1(A_{0,1})$, $j = 1, 2$. We then construct $l\varepsilon$-triangular caps

$T_{1,j}$ on $A_{1,j}$. We name the four new edges $A_{2,j}$, $j \in \{1,2,3,4\}$. We denote this second "approximation", $\cup_{j=1}^{2} T_{1,j}$ by A_1. We note that $A_{2,j}$, $j = 1, ..., 4$ are the 2^2 shortest edges of length l^2. We note also that A_1 can also be constructed by the appropriately rotated union of two copies of A_0

We now continue inductively, suppose that we have a set A_n consisting of 2^n triangular caps, $T_{n,j}$ with base length l^{n-1} and altogether 2^{n+1} "shortest sides", $A_{n+1,j}$ of length l^{n+1}. On each $A_{n+1,j}$ we construct a $l^{n+1}\varepsilon$-triangular cap $T_{n+1,j}$. We set

$$A_{n+1} := \bigcup_{j=1}^{2^{n+1}} T_{n+1,j}.$$

This A_{n+1} will then have all of the same properties as A_n with n replaced by $n+1$. We note also that with the numbering of the caps, we always count from "left" to "right" so that $T_{n+1,2j-1} \cup T_{n+1,2j} \subset T_{n,j}$.

We then define

$$\Gamma_\varepsilon = \bigcap_{n=0}^{\infty} A_n$$

where the dependence on ε comes from the initial choice of ε. $\diamondsuit$

One property of Γ_ε that should be noted now, as it is particularly intrinsic to the construction is that Γ_ε is essentially the union of two scaled copies of itself. We show this after the following definitions.

Definition 3.2.
We denote the end points of a line of finite length, A, as $E(A)$, and call them the **edge points of** *A. Let $T_{n,i}$ be a triangular cap. $T_{n,i}$ will then have 3 vertices which will be called the* **edge points of** *T. Let A_n be a stage in Construction 1 or 2 (we will see that the definition applies to definition as well to Construction 2) then the* **edge points of** *A_n are*

$$E(A_n) := \bigcup_{i=1}^{2^n} E(T_{n,i})$$

and the **edge points of** *Γ_ε are*

$$E(\Gamma_\varepsilon) := \bigcup_{n=1}^{\infty} E(A_n).$$

Also, as we will see, the same definition applies to A_ε. That is we can and do define the **edge points of** A_ε *to be*

$$E(A_\varepsilon) := \bigcup_{n=1}^{\infty} E(A_n).$$

We see the the edge points are all of the corners that appear in the constructions of Γ_ε and A_ε.

Definition 3.3.
We define the **edgeless** Γ_ε *as*

$$\Gamma_\varepsilon^E := \Gamma_\varepsilon \sim E(\Gamma_\varepsilon).$$

Proposition 3.1.
There are contraction mappings, S_1 and S_2, and an open set, O, such that

$$\Gamma_\varepsilon^E \subset O,$$

$$S_1(O) \cup S_2(O) \subset O,$$
$$S_1(\Gamma_\varepsilon^E) \cup S_2(\Gamma_\varepsilon^E) = \Gamma_\varepsilon^E$$

and

$$S_1(O) \cap S_2(O) = \emptyset.$$

Further

$$\begin{aligned} LipS_1 = LipS_2 &= l \\ &:= (1/4 + \varepsilon^2)^{1/2} \end{aligned}$$

Proof:
It is not too difficult to check that the contraction mappings of Lipschitz constants l defined by

$$S_i(x,y) = \begin{pmatrix} cos((-1)^i tan^{-1}(\varepsilon) - \pi) & -sin((-1)^i tan^{-1}(\varepsilon) - \pi) \\ sin((-1)^i tan^{-1}(\varepsilon) - \pi) & cos((-1)^i tan^{-1}(\varepsilon) - \pi) \end{pmatrix} v(x,y)$$

where

$$v(x,y) = \left(l \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1/2 \\ \varepsilon/2 \end{pmatrix} \right) + \begin{pmatrix} 1/2 \\ \varepsilon/2 \end{pmatrix} \right)$$

are such that

$$S_1(T_0) = T_{1,2},$$

$$S_2(T_0) = T_{1,1}$$

and thus

$$S_1(A_0) \cup S_2(A_0) = S_1(T_0) \cup S_2(T_0) = T_{1,1} \cup T_{1,2} = A_1.$$

Further, by setting O to be the open quadrilateral with vertices

$$\{(0,0), (1/2, 3\varepsilon/2), (1,0), (1/2, -\varepsilon/2\}$$

we see

$$\Gamma_\varepsilon^E \subset T_0 = A_0 \subset O$$

and that we have $S_1(O)$ is the quadrilateral of vertices

$$\{(1/2,\varepsilon), (1,0), l((1,0)-(1/2,3\varepsilon/2))+(1/2,3\varepsilon/2), l((1,0)-(1/2,-\varepsilon/2))+(1/2,-\varepsilon/2)\}$$

and $S_2(O)$ is the quadrilateral of vertices

$$\{(0,0), l(1/2, 3\varepsilon/2), (1/2, \varepsilon), l(1/2, -\varepsilon/2)\}.$$

It follows that

$$S_1(O) \cup S_2(O) \subset O$$

and

$$S_1(O) \cap S_2(O) = \emptyset.$$

Note specifically that since the proceedure, P, of taking two triangular caps on the shorter sides of a union of isosceles triangles is clearly invariant under orthogonal transformation (since chosing the new cap to be within the previous triangle is independent of orientation) and homothety, that is $P(R(T)) = O(R(T))$ where T is an isosceles triangles and R is any orthogonal transformation on $\mathbb{R}^2$, and if $l \in \mathbb{R}$, $P(lT) = lP(T)$. Since S_1 and S_2 are indeed just combinations of homothety and orthogonal transformation we have $P(S_i(T)) = S_i(P(T))$ for $i = 1, 2$.

We claim that

$$A_n = S_1(A_{n-1}) \cup S_2(A_{n-1})$$

for each $n \in \mathbb{N}$.

We already have a starting point ($n = 1$). Now, supposing that

$$A_n = S_1(A_{n-1}) \cup S_2(A_{n-1})$$

for some $n \in \mathbb{N}$, we then have

$$\begin{aligned} A_{n+1} &= P(A_n) \\ &= P(S(A_{n-1}) \cup S_2(A_{n-1}) \\ &= S_1(PA_{n-1}) \cup S_2(PA_{n-1}) \\ &= S_1(A_n) \cup S_2(A_n) \end{aligned}$$

Completing the induction. Then, since $A_1 \subset A_0$, we then have

$$\begin{aligned} \Gamma_\varepsilon^E &= \bigcap_{n=0}^{\infty} A_n \sim E \\ &= \bigcap_{n=1}^{\infty} (A_n \sim E) \\ &= \bigcap_{n=1}^{\infty} S_1(A_{n-1} \sim E) \cup S_2(A_{n-1} \sim E) \\ &= \bigcap_{n=0}^{\infty} S_1(A_n \sim E) \cup S_2(A_n \sim E) \\ &= S_1\left(\bigcap_{n=1}^{\infty} A_n \sim E\right) \cup S_2\left(\bigcap_{n=1}^{\infty} A_n \sim E\right) \\ &= S_1(\Gamma_\varepsilon^E \sim E) \cup S_2(\Gamma_\varepsilon^E \sim E). \end{aligned}$$

$\diamondsuit$

3.2 Pseudo-Fractal Sets

We now construct the "strangest" sets. These are similar to Γ_ε in construction, however, as we noted in Proposition 3.1, the construction for Γ_ε retains the basic shape of the triangular caps. This will not be sufficient for the cases when we want to prove properties for the case where approximations should hold for all $\delta > 0$. We therefore allow the relative height of the triangular caps to shrink, so that the "angles" involved in the triangles approach zero as we look at smaller and smaller sections of the triangles. As we will see later, even this adjustment is not sufficient. We therefore remove all of the interior at each stage, take, in a sense, a limit, remove the approximating sets and the edges. We make the specific constructions below in Constructions 3.2 and 3.3. Note that the heuristic path to our set just given was not the one that originally led to its construction, but rather, it is the result of being the

embodiment of the worst case allowed in a failed attempt to prove that the answer to Definition A (iv) (2) was yes. As has been mentioned, the third set is then a carefully selected subset of this chosen in such a way as to ensure that it is closed.

Construction 3.2. .
We construct the set, as previously, as a subset of $\mathbb{R}^2$. We start with

$$A_0 := [(0,0),(1,0)].$$

We then denote by T_0 the 2ε-triangular cap on A_0.

We now set

$$A_1 := \overline{(\partial T_{0,1} \sim A_0)},$$

which is the union of two lines (namely the two shorter edge lines of $T_{0,1}$), we name the two lines $A_{1,i}$, $i = 1,2$. To continue, we denote by $T_{1,i}$ the ε-triangular cap constructed on $A_{1,i}$, considered as an edge of $T_{0,1}$ for each i.

We then set

$$A_2 := \overline{\left(\partial\left(\bigcup_{i=1}^{2} T_{1,i}\right) \sim A_1\right)},$$

which will be a union of 4 lines $A_{2,i}$, $i = 1,2,3,4$. Each an edge of a triangular cap $T_{1,i}$.

We continue the construction inductively. Assuming we have $\mathcal{A}_n$, a union of 2^n lines, $\{A_{n,i}\}_{i=1}^{2^n}$ that lie on the boundary of 2^{n-1} triangular caps $\{T_{n-1,i}\}_{i=1}^{2^{n-1}}$, (and A_n a union of 2^n triangular cups), we construct 2^n $2^{1-n}\varepsilon$-triangular caps, $\{T_{n,i}\}_{i=1}^{2^n}$, on each of the 2^n lines. As previously we number from "left" to "right" so that $T_{n+1,2j-1} \cup T_{n+1,2j} \subset T_{n,j}$. We then set

$$A_{n+1} := \overline{\left(\partial\left(\bigcup_{i=1}^{2^n} T_{n,i}\right) \sim A_n\right)},$$

Finally, we define

$$\begin{aligned} A_\varepsilon &:= \bigcap_{i=1}^{\infty}\left(\overline{\bigcup_{n=1}^{\infty} A_n} \sim \bigcup_{n=1}^{i}(A_n \sim E)\right) \sim E \\ &= \overline{\bigcup_{n=1}^{\infty} A_n} \sim \bigcup_{n=1}^{\infty} A_n, \end{aligned}$$

where

$$E = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} E(A_{n,i}),$$

and $E(A_{n,i})$ denotes the endpoints of the line $A_{n,i}$. As previously, the ε refers to the arbitraryily chosen $\varepsilon > 0$ at the begining of the construction, which may, of course, be chosen, as small as is necessary. $\diamondsuit$

Remarks:

(1) The removal of the endpoints is very important for the example. With the endpoints, there are of course points in the set with a fixed angle that must be squeezed into a δ approximation for exery $\delta > 0$. This is not possible. With the endpoints missing we can, for each element of the set choose, for any given angle greater than zero, avoid all "corners" of angle greater than or equal to the given one, so as to make the set flater than the given angle in that neighbourhood. Since we are asking questions of measure, it is also important to note that the union of all the endpoints, that is

$$\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2^n} E(A_{n,i}),$$

is countable and therefore of zero $\mathcal{H}^1$ measure, thus having no effect on any $\mathcal{H}^1$ measure properties that we are looking at.

(2) A remark on both Construction 3.1 and Construction 3.2 and indeed on Definition 3.1 is that a triangular cap constructed on the edge of a previously constructed triangular cap may not be well defined in that it may not be a subset of the previous triangular cap. Another problem is that, as we often do, constructing triangular caps on both of the sides of identical length on an isoceles triangle may lead to the two new triangular caps indeed being subsets of the previous cap, but intersecting with each other. Choosing the vertical heights prevents this problem, and indeed, should the initial vertical heigth, h be less than 1/4 of the base length b, then provided the new vetical height is less than or equal to $h(\sqrt{h^2 + b^2/4})/b$ we will encounter no problems, such a proceedure cannot lead to a new vertical height being more than 1/4 the base length, and furthermore in this situation, $h(\sqrt{h^2 + b^2/4})/b = l > h/2$ (where the l is the scaling factor in Construction 3.1) so that no problems later in the induction can occur in Constructions 3.1, 3.2 and 3.3. We will always assume that the appropriate conditions on the vertical height have been satisfied. This is no problem as we want our caps to be very flat in any case.

Definition 3.4.
For each $n \in \mathbb{N} \cup \{0\}$ and each $i \in \{1, ..., 2^n\}$ there is a triangular cap $T_{n,i}$ constructed on $A_{n,i}$, we denote the vertice of $T_{n,i}$ that is not in $A_{n,i}$ (that is, the new vertice created) by $a_{n,i}$.

Construction 3.3. .
As previously mentioned we will be looking at a subset of A_ε. We have already noted that the edge points of A_ε are countable, we now give them an ordering that will prove important later. We take

$$e_1 = (0, 0),$$

$$e_2 = (1, 0),$$

$$e_3 = a_{0,1}$$

and then in general

$$e_{2+i+\sum_{j=0}^{n-1} 2^j} = a_{n,i}.$$

We set

$$\rho_1 = \frac{1}{4} 2^{-7} (1 + 7 \cdot 16\varepsilon^2)^{1/2}$$

and for $n \in \mathbb{N}$, we set

$$\rho_n = 4^{1-n} r_1 < 2^{-6-n} (1 + (n + 6)16\varepsilon^2)^{1/2}.$$

We now define a set of radii. We set

$$r_1 = \rho_1$$

and

$$r_2 = \rho_2.$$

Then for $i \geq 2$ there is a unique $n \in \mathbb{N} \cup \{0\}$ such that

$$i \in \{2 + \sum_{j=0}^{n} 2^n, ..., 1 + \sum_{j=0}^{n+1} 2^j\}$$

so that we can define

$$r_i = \min\{\rho_i, d(e_i, A_{n-1} \cup A_{n-2})/2\}.$$

We then define

$$\mathcal{B} := \bigcup_{i=1}^{\infty} B_{r_i}(e_i)$$

Note that

$$E = \{e_i\}_{i=1}^{\infty} \subset \bigcup_{i=1}^{\infty} B_{r_i}(e_i) = \mathcal{B}.$$

Finally we define

$$\mathcal{A}_\varepsilon = A_\varepsilon \sim \mathcal{B}.$$

We note that this can also be written as

$$\mathcal{A}_\varepsilon = (A_\varepsilon \cup E) \sim \mathcal{B}.$$

$\diamondsuit$

There are three points concerning A_ε and $\mathcal{A}_\varepsilon$ that are important that should be noted. Firstly, the entire purpose of altering A_ε to $\mathcal{A}_\varepsilon$ was that $\mathcal{A}_\varepsilon$ should be closed. We therefore prove that this important property indeed holds. Secondly, although we will show that A_ε and $\mathcal{A}_\varepsilon$ have property (iv) with respect to $j = 1$ and thus have dimension 1, the sets have some interesting properties in and of themselves. For this reason and as support for the consistency of the results here we provide a direct proof that the dimension of A_ε and $\mathcal{A}_\varepsilon$ is 1. Finally, as we will show in chapter 4, the exotic counterexamples of A_ε and $\mathcal{A}_\varepsilon$ are necessary. Further, to support the idea that counter examples to (iv) 2 need necessarily be badly behaved, we note that A_ε, $\mathcal{A}_\varepsilon$ are not rectifiable. As substantial preparation is necessary and since the fact is not necessary for our classification, we present the proof in Chapter 7 along with the generalisations of the sets. A direct proof for these specific examples is also given.

Lemma 3.1.
$\mathcal{A}_\varepsilon$ is closed.

Proof:
We first show that $A_\varepsilon + E$ is closed.

Consider a convergent sequence of points $\{x_n\} \subset A_\varepsilon + E$. We must show that

$$x := \lim_{n\to\infty} x_n \in A_\varepsilon + E.$$

If

$$x \in E$$

we are finished, so assume that this is not the case. Now, for each x_n, either $x_n \in E$ or $x_n \in A_\varepsilon$.

In the first case $x_n = e_i \in E$ and there is an $n_0 \in \mathbb{N}$ such that $e_i \in A_n$ for each $n > n_0$. By taking $\{x_{n,j}\}_{j=1}^{\infty}$ such that $x_{n,j} = x_n$ for each j then $x_{n,j} \in A_m$ for some $m \geq j$ for each j and $\lim_{j\to\infty} x_{n,j} = x_n$.

In the second case

$$x_n \in A_\varepsilon = \overline{\left(\bigcup_{m=1}^{\infty} A_m\right)} \sim \bigcup_{m=1}^{\infty} A_m.$$

Thus there exists a sequence $x_{n,j}$ such that $|x_{n,j} - x_n| < 1/j$ so that $\lim_{j\to\infty} x_{n,j} = x_n$ and $\{x_{n,j}\}_{j=1}^{\infty} \subset \bigcup_{m=1}^{\infty} A_m$. Now assume that there is a finite number q such that $\{x_{n,j}\}_j \subset \cup_{m=1}^{q} A_m$. Then since $\cup_{m=1}^{q} A_m$ is a finite union of closed lines it is closed so that $\lim_j x_{n,j} \in \cup_{m=1}^{q} A_m$ and thus $x_n \in \cup_{m=1}^{q} A_m$. However, since $x \notin E$ and $\cup_{m=1}^{q} E(A_m)$ is finite, $d(x, \cup_{m=1}^{q} E(A_m) > 0$. Thus in this case $x_n \in \cup_{m=1}^{q} A_m \sim E$. It follows then that we would have

$$\begin{aligned} x_n &\notin \overline{\left(\bigcup_{m=1}^{\infty} A_m\right)} \sim \bigcup_{m=1}^{\infty} A_m + E \\ &= A_\varepsilon \cup E. \end{aligned}$$

We can therefore take a subsequence and relabel to assume that $x_{n,j} \in A_m$ for some $m \geq j$ for each $j \in \mathbb{N}$.

We now take the sequence $\{x_m\}_{m=1}^{\infty}$ given by

$$x_m = x_{m,m},$$

and note that $\{x_m\} \subset \bigcup_{n=1}^{\infty} A_n$ so that $\lim_{m\to\infty} x_m \in \overline{\cup_{n=1}^{\infty} A_n}$. By the condition that $|x_{n,j} - x_n| < 1/j$, this diagonal selection gives us

$$\begin{aligned} x &= \lim_{m\to\infty} x_m \\ &\in \overline{\bigcup_{n=1}^{\infty} A_n}. \end{aligned}$$

Since, following from construction 2, for each $n \in \mathbb{N}$ and each $y \in A_n \sim E$ there is a radius $r > 0$ such that $d(y, \cup_{m=n+1}^{\infty} A_m) > r$ it follows that for each $n \in \mathbb{N}$ $x \notin A_n \sim E$. Thus

$$\begin{aligned} x &\in \overline{\left(\bigcup_{m=1}^{\infty} A_m\right)} \sim \bigcup_{m=1}^{\infty} (A_m \sim E) \\ &= \overline{\left(\bigcup_{m=1}^{\infty} A_m\right)} \sim \bigcup_{m=1}^{\infty} A_m \cup E \\ &= A_\varepsilon \cup E. \end{aligned}$$

We therefore have that A_ε is closed.

Now since $\mathcal{B}$ is the countable union of open balls it is also open. Since $E \subset \mathcal{B}$ we can write

$$\mathcal{A}_\varepsilon = A_\varepsilon \sim \mathcal{B} = A_\varepsilon \cup E \sim \mathcal{B}$$

which is a closed set without an open set and thus is closed, proving the Lemma. $\diamondsuit$

3.3 Properties of A_ε and $\mathcal{A}_\varepsilon$

We now look at some direct properties of A_ε and $\mathcal{A}_\varepsilon$ that will be important to us later. Some of the properties, for example the dimension of A_ε and $\mathcal{A}_\varepsilon$ follow from more general Theorems that we shall use. However, since the direct proof is more instructive as to the properties of the sets and is not particularly longer, we present the direct proof here.

Lemma 3.2.
Let $\varepsilon > 0$ be such that A_ε is well defined. Then for each $n \in \mathbb{N}$ and each $j \in \{1, 2, ..., 2^n\}$, the base length of a triangular cap $T_{n,j}$ in the construction of A_ε has length

$$\mathcal{H}^1(A_{n,j}) = \frac{(1 + n16\varepsilon^2)^{1/2}}{2^n},$$

and thus

$$\mathcal{H}^1(A_n) = (1 + n16\varepsilon^2)^{1/2}$$

for each $n \in \mathbb{N}$.

Proof:
Clearly $\mathcal{H}^1(A_0) = 1$.

$$\mathcal{H}^1(A) \geq \liminf_{n\to\infty} \mathcal{H}^1(A_n).$$

Then $\mathcal{H}^1(A_1)$ is the sum of two hypotheses of triangles $(1/2)\mathcal{H}^1(A_0)$ base length and 2ε height. that is

$$\begin{aligned}\mathcal{H}^1(A_1) &= 2\left(\left(\frac{1}{2}\right)^2 + (2\varepsilon)\right)^{1/2} \\ &= (1 + 16\varepsilon^2)^{1/2}\end{aligned}$$

Having that it is true for $n = 0, 1$ I now claim that

$$\mathcal{H}^1(A_n) = (1 + n16\varepsilon^2)^{1/2}.$$

Assuming it is true for n we note that $\mathcal{H}^1(A_{n+1})$ is the sum of 2^{n+1} hypotheses of triangles of base length $\mathcal{H}^1(A_n)/2^{n+1}$ and height $2^{2-n}\varepsilon$. That is

$$\begin{aligned}
\mathcal{H}^1(A_{n+1}) &= 2^{n+1}\left(\left(\frac{\mathcal{H}^1(A_n)}{2^{n+1}}\right)^2 + (2^{2-(n+1)}\varepsilon)^2\right)^{1/2} \\
&= ((\mathcal{H}^1(A_n))^2 + 2^{2-2n+2+2n}\varepsilon)^{1/2} \\
&= (1 + n16\varepsilon^2 + 16\varepsilon^2)^{1/2} \\
&= (1 + (n+1)16\varepsilon^2)^{1/2},
\end{aligned}$$

proving the inductive claim. Then for each $n \in \mathbb{N}$ and $j \in \{1, 2, ..., 2^n\}$ the base length of a triangular cap $T_{n,i}$ is one equal 2^n-th part of the length of A_n. That is

$$\mathcal{H}^1(A_{n,j}) = \frac{(1 + n16\varepsilon^2)^{1/2}}{2^n}.$$

$\diamondsuit$

Definition 3.5.
We denote the projection of a space onto a subset, S, whenever thye concept of projection makes sense for S by π_S. An exception to this rule is the projection of $\mathbb{R}^2$ onto the x-axis identified with $\mathbb{R}$. This projection is denoted by π_x.

Theorem 3.1.
$dim A_\varepsilon = dim \mathcal{A}_\varepsilon = 1.$

Proof:
First note that

$$\mathcal{H}^1(A_\varepsilon) \geq \mathcal{H}^1(\pi_x(A_\varepsilon))$$

and similarly

$$\mathcal{H}^1(\mathcal{A}_\varepsilon) \geq \mathcal{H}^1(\pi_x(\mathcal{A}_\varepsilon))$$

First, since E is countable we can consider $x \in [0,1] \sim \pi_x(E) \neq \emptyset$. Since each A_n can be considered as a connected path joining $(0,0)$ and $(0,1)$ there is an $x_n \in \pi_x^{-1}(x) \cap A_n$. Then we have $\{x_n\}_n$ a subsequence of $\cup_{n=1}^{\infty} A_n$. Since this sequence is in a bounded set ($[0,1] \times [0, 2\varepsilon]$) there is a convergent subsequence. Since for all $n \in \mathbb{N}$, $\pi_x(x_n) \notin \pi_x(E)$, it follows that $x_0 = \lim x_n \notin E$. Similarly to in the previous Lemma, this also implies that $x \notin A_m$ for each $m \in \mathbb{N}$. Therefore

$$x_0 \in \overline{(\cup_{n=1}^{\infty} A_n)} \sim \cup_{n=1}^{\infty A_n} \sim E = A_\varepsilon.$$

It follows that

$$\begin{aligned}
\mathcal{H}^1(\pi_x(A_\varepsilon) &\geq \mathcal{H}^1([0,1] \sim \pi_x(E)) \\
&\geq \mathcal{H}^1([0,1]) - \mathcal{H}^(\pi_x(E)) \\
&= 1 \\
&> 0.
\end{aligned}$$

Now, we note that $\mathcal{A}_\varepsilon = A_\varepsilon - \mathcal{B}$ and that

$$r_i = \frac{1}{4}2^{-7}(1 + 7 \cdot 16\varepsilon^2)^{1/2} < 2^{-7}$$

(since we are in any case always taking $\varepsilon < 0.01$). It follows that

$$\begin{aligned}
\mathcal{H}^1(\pi_x(\mathcal{A}_\varepsilon)) &= \mathcal{H}^1(\pi_x(A_\varepsilon \sim \pi_x(\mathcal{B})) \\
&\geq \mathcal{H}^1(A_\varepsilon) - \sum_{i=1}^{\infty} r_i \\
&> 1 - 2^{-7}\sum_{i=1}^{\infty} 4^{-i} \\
&> 1 - 2^{-7} \\
&> 0.
\end{aligned}$$

It follows that

$$\dim A_\varepsilon \geq \dim \mathcal{A}_\varepsilon \geq 1 \tag{3.1}$$

Now let $s > 0$ and $\delta > 0$. Then for any given $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that

$$\delta \in (2^{2-n}\varepsilon, 2^{3-n}\varepsilon]. \tag{3.2}$$

We note that the vertical height of the trianglular caps in the n-th stage of construction of A_ε is 2^{1-n} so that $\delta \geq 2$ times the vertical height of the triangular caps in the nth construction stage. Since

$$A_\varepsilon \subset \bigcup_{i=1}^{2^n} T_{n,i}$$

any cover of $\cup_{i=1}^{2^n}$ is also a cover of A_ε. By taking balls of radius δ with centers in A_n we note that we can take these balls along an $A_{n,i}$ such that

the overlaps ensure that $A_{n,i}^{\delta/\sqrt{2}}$ is covered. By taking such a cover of $A_{n,i}$ for each i we have a cover consisting of balls of radius δ, $\mathcal{B}_\delta = \{B_\delta\}$ such that

$$\bigcup_{B_\delta \in \mathcal{B}_\delta} B_\delta \supset A_n^{\delta/\sqrt{2}} \supset A_n^{2^{1-n}}.$$

Since

$$\begin{aligned} A_n^{2^{1-n}} &\supset \bigcup_{i=1}^{2^n} T_{n,i} \\ &\supset A_\varepsilon \end{aligned}$$

we also have that $\mathcal{B}_\delta$ is a cover of A_ε. Since with such a cover no more than $\delta/\sqrt{2}$ of the radius of a ball in $\mathcal{B}_\delta$ will uniquely contribute to the cover of A_n, and since the inefficiencies of taking $A_{n,i}$'s that meet at non-uniform angles can not do any worse than forcing us to cover A_n twice it follows that

$$\sum_{B_\delta \in \mathcal{B}_\delta} \delta \leq 2\sqrt{2}\mathcal{H}^1(A_n)$$

so that from Lemma 5 we have

$$\sum_{B_\delta \in \mathcal{B}_\delta} \delta \leq 2\sqrt{2}(1 + n16\varepsilon^2)^{1/2}.$$

Thus from (3.2) we have

$$\sum_{B_\delta \in \mathcal{B}_\delta} \delta^{1+s} \leq (2\varepsilon)^s 2\sqrt{2}\frac{(1 + n16\varepsilon^2)^{1/2}}{2^{ns}}.$$

Since $\mathcal{B}_\delta$ is a cover of A_ε this means

$$\mathcal{H}_\delta^{1+s}(A_\varepsilon) \leq (2\varepsilon)^s 2\sqrt{2}\frac{(1 + n16\varepsilon^2)^{1/2}}{2^{ns}}$$

so that we have

$$\begin{aligned} \mathcal{H}^{1+s}(A_\varepsilon) &= \lim_{\delta \to 0} \mathcal{H}_\delta^{1+s} \\ &\leq \lim_{n \to \infty} (2\varepsilon)^s 2\sqrt{2}\frac{(1 + n16\varepsilon^2)^{1/2}}{2^{ns}} \\ &= 0. \end{aligned}$$

Since this is true for all $s > 0$ it follows that $\dim A_\varepsilon \leq 1$ and since $\mathcal{A}_\varepsilon \subset A_\varepsilon$ that $\dim \mathcal{A}_\varepsilon \leq 1$. Combining with (3.1) gives the result. $\diamondsuit$

Chapter 4

The limited Potency of Simple Examples and Weak Requirements for Locally Finite Measure

4.1 Limits on Approximately j-Dimensional Sets Entering and Exiting on the Same Side

As we have already mentioned, several of the questions we are asking must be answered in the negative. To show this, clearly we need counter examples. Some of the counter examples, such as $\mathcal{N}$, Λ_δ and Λ^2 are relatively simple in that they are countable collections of nicely behaved functions whose relevant properties are clear. Γ_ε is not so transparent as the sets already mentioned. It is, however, relatively clear that we need something a bit more complex to satisfy a j-dimensional approximation with a set that is not j-dimensional so as to provide a counter example for those properties not ensuring j-dimensionality.

A_ε and $\mathcal{A}_\varepsilon$, however are another matter, being "pseudo-fractal" sets (in the sense that every magnification of A_{ε_1} looks like A_{ε_2} for some $\varepsilon_2 < \varepsilon_1$ so that A_ε is semi-selfsimilar.) that are in fact j-dimensional (where $j = 1$ in this case). The obvious question is to ask if we could find a tricky way of putting nicely behaved functions together to get a different counter example to (iv) (2). (iv) is particularly important as we actually know that some singularity sets with a relationship to this property. We answer this question with an encouraging "no". This is encouraging as it means that to show that singu-

larity sets have any sort of nice properties would then directly imply locally finite $\mathcal{H}^j$- measure. In fact, as mentioned previously, we can show that A_ε and $\mathcal{A}_\varepsilon$ are not countably j-rectifiable for the j used in property (iv) which, since A_ε and $\mathcal{A}_\varepsilon$ are the only known counterexamples certainly supports the assertion that such sets must be poorly behaved.

We find that any counter example must in fact be very poorly behaved in that for any point of locally infinite measure (where the essential part of a counter example is) cannot possibly have any part of the set (no matter how small) going through it that could be almost everywhere described by a Lipschitz function under some rotation and still satisfy property (iv). That is the set has to be a broken non-function at all critical points at all magnifications.

Conversely this means, to ensure a singularity set satisfying (iv) is locally $\mathcal{H}^j$ finite we would expect only to need to show that no point on the singularity set has a neighbourhood in which the singularity set is purely unrectifiable.

This section proves these assumptions. The key idea is that to have a function of infinite measure in a small neighbourhood means that at some point it has to be sharply folded on itself at all levels of magnification which will prevent the set from having property (iv). We make a couple of necessary definitions, then prove a Lemma proving an important special case which we use in the Theorem proving our claim.

Definition 4.1.
Let $u : \mathbb{R} \to \mathbb{R}$ be a function and let

$$graph u \cap B_\rho(y) \subset L_y^\delta$$

for some affine space L_y and some $\delta \in (0, 1/4)$. Then u is said to **enter and exit the same side of $B_\rho(y)$ with respect to L_y^δ** *if*

$$\max\{|z - x| : y, x \in graph u \cap \partial B_\rho(y)\} < \frac{\pi\rho}{2}.$$

We note then that for a ball $B_\rho(y)$ and an affine space $L_y \ni y$

$$L_y^\delta \cap \partial B_\rho(y) = \Psi_1 \cup \Psi_2$$

for some arcs Ψ_1 and Ψ_2 in $\mathbb{R}^2$. We can therefore make the following definition.

Definition 4.2.
Suppose a function u enters and exits $B_{\rho(y)}$ on the same side with respect to L_y^δ. Then

$$L_y^\delta \cap \partial B_\rho(y) = \Psi_1 \cup \Psi_2$$

for some arcs Ψ_1 and Ψ_2 in $\mathbb{R}^2$. Further graph$u \cap \Psi_i \neq \emptyset$ for exactly one $i = i(u) \in \{1,2\}$. We denote this $\Psi_{i(u)}$ by Ψ_u and the other by Ψ^u.

Lemma 4.1.
Suppose $u : \mathbb{R} \to \mathbb{R}$ is continuous and graph$u \subset A \subset \mathbb{R}^2$. Suppose that A has property (iv) and that for some $y \in A$ and $\delta \in (0, 1/4)$ ρ_y is an appropriate radius at y with respect to δ. If u enters and exits $B_{\rho_y}(y)$ on the same side. Then

$$\max\{d(\Psi_u, y) : y \in \textit{graph}u \cap B_{\rho_y}(y)\} < 4\delta\rho_y.$$

Proof:
We first show that

$$\text{graph}u \cap B_{\rho_y}(y) \subset \text{graph}u(I_{u,y,\rho_y})$$

where

$$I_{u,y,\rho_y} := [\inf\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}, \sup\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}].$$

Suppose that this were not to be the case, then there is a $z \in B_{\rho_y}(y) \subset \mathbb{R}^2$ with $z \in$ graph$u$ (and thus $u(\pi_x(z)) = z$) and such that either

$$\pi_x(z) > \sup\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}\text{or}$$

$$\pi_x(z) < \inf\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}.$$

without loss of generality we consider the case

$$\pi_x(z) > \sup\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}$$

the other case follows similarly. Since u is a continuous function graph$u$ is connected and by the choice of z

$$\max\{\pi_x(B_{\rho_y}(y))\} > \max\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y)\}$$

Thus the path

$$P := u([\max\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}, \max\{\pi_x(B_{\rho_y}(y))\} + 1])$$

intersects $B_{\rho_y}(y)$ only at its starting point on the boundary of $B_{\rho_y}(y)$. That is

$$P \cap B_{\rho_y}(y) = u(\max\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y)\})$$

(Otherwise $u(x) \cap \partial B_{\rho_y}(y) \neq \emptyset$ for some $x > \max\{\pi_x(\text{graph}u \cap \partial B_{\rho_y}(y))\}$ (in order for the connected path, P, to leave the ball) contradicting the choice of $\max\{\pi_x(\text{ graph}u \cap \partial B_{\rho_y}(y))\}$.)

Thus

$$\pi_x(z) \in \left[\max\{\pi_x(\text{ graph}u \cap \partial B_{\rho_y}(y))\}, \max\{\pi_x(B_{\rho_y}(y)\} + 1\right]$$

which implies

$$u(\pi_x(z)) \notin B_{\rho_y}(y).$$

This contradiction means that $z \notin$ graph$u$.

For $z \in$ graph$u \cap B_{\rho_y}(y)$ Let

$$z_\partial := \pi_x^{-1}(\pi_x(z)) \cap \Psi_u$$

which will be a unique point. Now assume

$$\max\{d(\Psi_u, z) : z \in \text{graph}u \cap B_{\rho_y}(y)\} \geq 4\delta\rho_y$$

Then there is a $z \in$ graph$u \cap B_{\rho_y}(y)$ such that

$$\begin{aligned} |\pi_y(z) - \pi_y(z_\partial)| &> d(z, z_\partial) \\ &> 4\delta\rho_y, \end{aligned}$$

since for all $a \in \Psi_u$, $|a - z_\partial| < 2\delta\rho_y$ and thus $|\pi_y(a) - \pi_y(z_\partial)| < 2\delta\rho_y$.

This implies

$$\inf\{|\pi_y(z) - \pi_y(a)| : a \in \Psi_u\} > 2\delta\rho_y.$$

w.l.o.g. assume that $\pi_y(z) > \sup\{\pi_y(a) : a \in \Psi_u\}$.

Then, as u is continuous, there exist two connected paths P_1, P_2 such that

$\pi_x(P_1) \leq \pi_x(z)$,

$\pi_x(P_2) \geq \pi_x(z)$ and

P_1 and P_2 are connected to Ψ_u.

Thus
$$P_1 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y) \neq \emptyset$$
and
$$P_2 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y) \neq \emptyset.$$
Let
$$z_1 \in P_1 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y)$$
and
$$z_2 \in P_2 \cap \pi_y^{-1}(\pi_y(z) - 2\delta\rho_y).$$
Without loss of generality assume $|\pi_x(z_1) - \pi_x(z)| \leq |\pi_x(z_2) - \pi_x(z)|$ This choice implies that

$$\begin{aligned}
|\pi_x(z_1) - \pi_x(z)| &\leq 1/2 \sup\{|\pi_x(a_1) - \pi_x(a_2)| : a_1, a_2 \in \Psi_u\} \\
&\leq \delta\rho_y.
\end{aligned}$$

Then notice

$$\begin{aligned}
\rho_z &:= |z_2 - z_1| \\
&\leq \sup\{|\pi_x(a_1) - \pi_x(a_2)| : a_1, a_2 \in \Psi_u\} \\
&= 2\delta\rho_y \\
&\leq 1/2\rho_y
\end{aligned}$$

so we consider $B_{5\rho_z/4}(z_1)$.

Notice also that $|\pi_x(z) - \pi_x(z_1)| < |\pi_x(z_2) - \pi_x(z)|$ implies

$$|\pi_x(z) - \pi_x(z_1)| \leq \frac{1}{2}\rho_z.$$

Now call the subpath of $P_1 \subset \text{graph} u$ connecting z_1 to z P_{z_1}. Note

$$\pi_x(P_{z_1}) \subset [\pi_x(z_1), \pi_x(z)] \text{ and}$$

$$z \notin B_{\rho_z}(z_1).$$

which implies
$$P_{z_1} \cap \partial B_{\rho_z}(z_1) \neq \emptyset$$
and for all
$$w \in |\pi_x(w) - \pi_x(z_1)| < \frac{1}{2}\rho_z \text{ and}$$
$$d(w, z_1) = \rho_z$$

which implies

$$|\pi_y(w) - \pi_y(z_1)| > \frac{\sqrt{3}}{4}\rho_z.$$

However, for any choice of $L^{\delta\rho_z}_{z_1,\rho_z}$ we must have

$$\sup\{|\pi_y(l) - \pi_y(z_1)| : l \in L^{\delta\rho_z}_{z_1,\rho_z}\} < \frac{9}{4}\delta\rho_z.$$

Since $\delta < \frac{1}{16}$ we note

$$\frac{\sqrt{3}}{4}\rho_z > \frac{\rho_z}{4} > \frac{9\rho_z}{64} > \frac{9}{4}\delta\rho_z.$$

Thus it is impossible to choose a L_{z,ρ_z} such that

$$A \cap B_{\rho_z}(z_1) \subset L^{\delta\rho_z}_{z,\rho_z}.$$

This would imply A does not have property (iv). This contradiction proves the Lemma. $\diamondsuit$

4.2 Set Constraints for Dually Approximately j-Dimensionality and Infinite Density

We now prove the main theorem of this chapter by showing that we can reduce the problem to an application of the above lemma.

Theorem 4.1.
Suppose $A \subset \mathbb{R}^2$ and that there exists a $y \in A$ such that

$$\mathcal{H}^1(y \cap B_\rho(A)) = \infty \text{ for all } \rho > 0$$

and for some $\rho_1 > 0$,

$$y \in G_y^{-1}(graphu) \cap B_{\rho_1}(y) \text{ and}$$

$$\overline{B_{\rho_y}(y)A \cap G_y^{-1}(graphu)} = G_y^{-1}(graphu) \cap B_{\rho_y}(y)$$

where u is Lipschitz, $G_y \in G(1,2)$ and $G_y(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the rotation such that $G_y(G_y) = \mathbb{R}$.

Then A does not have property (iv) for $j = 1$.

Proof:
By the invariance of the relevant quantities under orthogonal transformations we can assume that $y = (0, 0)$ and $G_y = \mathbb{R}$.

Assume that A does satisfy satisfy property (iv).

Then for a given $\delta < 1/8$ there is a $\rho_y = \rho_y(y) \in (0, \rho_1)$ such that there exists an affine space L_{y,ρ_y} such that

$$A \cap B_{\rho_y}(y) \subset L_{y,\rho_y}^{\delta\rho_y}$$

and furthermore, for each $x \in A \cap B_{\rho_y}(y)$ and $\rho \in (0, \rho_y]$ there is an affine space $L_{x,\rho}$ such that

$$A \cap B_\rho(x) \subset L_{x,\rho}^{\delta\rho}.$$

Noting that $y \in$graph$u$ and that clearly

$$\begin{aligned} d(y, \partial B_{\rho_y}(y)) &= \rho_y \\ &> 4\delta\rho_y \end{aligned}$$

it follows that

$$\max\{d(\Psi_u, y) : y \in \text{graph}u \cap B_{\rho_y}(y)\} < 4\delta\rho_y$$

and thus by Lemma 4.1 u cannot enter and exit B_{ρ_y} on the same side with respect to any affine space.

In particalar for each $w \in L_{y,\rho_y}$

$$\text{graph}u \cap \pi^{-1}_{L_{y,\rho_y}}(w) \cap L_{y,\rho_y}^{\delta\rho_y} \neq \emptyset.$$

Also, if

$$A \cap B_{\rho_y/2}(y) \subset \text{ graph}u$$

then

$$\mathcal{H}^1(A \cap B_{\rho_y/2}(y)) \leq \frac{\rho_y}{2} \cdot \omega_1 \cdot \text{ Lip}u < \infty,$$

a contradiction to our assumptions on the measure of balls around y.

It follows that there exists an $x \in A \cap B_{\rho_y/2}(y)$ such that $x \notin$graph$u$.

Note that $\pi^{-1}_{L_{y,\rho_y}}(x) \cap$graph$u \neq \emptyset$ which implies

$$d(x, \text{ graph}u) \leq 2\delta\rho_y < \frac{1}{2}\rho_y.$$

Now select $z \in$graph$u$ such that

$$\begin{aligned} d(z,x) &< \frac{9}{8}\inf\{d(w,x) : w \in \text{ graph}u\} \\ &=: \frac{9}{8}d \\ &< \rho_y. \end{aligned}$$

By the hypotheses there is an $z_1 \in$graph$u \cap A \cap B_{(1/16)d}(z)$. We now consider $B_{\rho_x}(z_1) \ni x$.

Note that for any choice of L_{z_1,ρ_x}

$$L^{\delta\rho_x}_{z_1,\rho_x} \cap \partial B_{\rho_x}(z_1) = \Psi_1 \cup \Psi_2,$$

a union of two arcs as considered in Definition 4.2 and that

$$d(x, \partial B_{\rho_x}(z_1)) < \frac{1}{4}d.$$

This implies that for some $i = i(x) \in \{1,2\}$

$$\begin{aligned} \Psi_i &\subset B_{(1/4)d+2\delta\rho_x}(x) \\ &= B_{(1/4)d+2\delta(9/8)d}(x). \end{aligned}$$

Since δ was chosen such that $\delta < 1/8$

$$\begin{aligned} \frac{1}{4}d + 2\delta\frac{5}{4}d &< \frac{4}{16} + \frac{5}{16}d \\ &< \frac{15}{16}d \end{aligned}$$

which implies

$$\text{graph}u \cap \Psi_{i(x)} = \emptyset.$$

This in turn implies that u enters and exits $B_{\rho_x}(z_1)$ on the same side with respect to any affine space possibly allowing property (iv) to hold.

Since $z_1 \in$ graph$u$

$$\begin{aligned} \max\{d(w, \partial B_{\rho_x}(z_1)) : w \in \text{ graph}u\}) &= \rho_x \\ &> 4\delta\rho_x. \end{aligned}$$

This implies, by Lemma 4.1, that A does not have property (iv). This contradiction completes the proof of the Theorem. $\diamondsuit$

In order to more definitely relate what has previously been discussed to this result, I observe the following trivial corollaries.

Corollary 4.1.
Suppose $A \subset \mathbb{R}^2$ and that there exists a $y \in A$ such that

$$\mathcal{H}^1(y \cap B_\rho(A)) = \infty \text{ for all } \rho > 0$$

and for some $\rho_1 > 0$,

$$A \cap B_{\rho_1}(y) = G_y^{-1}\left(\bigcup_{n=1}^{Q} graph u_n\right) \cap B_{\rho_1}(y)$$

for some $Q \in \mathbb{N} \cup \{\infty\}$ where u_n is Lipschitz for each n, $G_y \in G(1,2)$ and $G_y(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the rotation such that $G_y(G_y) = \mathbb{R}$.

Then A does not have property (iv).

Proof:
Since

$$y \in A \cap B_{\rho_1}(y) = G_y^{-1}\left(\bigcup_{n=1}^{Q} \text{graph} u_n\right) \cap B_{\rho_1}(y)$$

$y \in \text{graph} u_{n_0}$ for some $1 \leq n_0 \leq Q$. With $u = u_{n_0}$ the conditions of Theorem 2 are then satisfied from which the conclusion follows. $\diamondsuit$

Corollary 4.2.
$\mathcal{N}$, Λ_δ and Λ^2 are not counter examples to (iv) (2).

Proof:
Let $\Xi = \mathcal{N}$ or Λ_δ. Then since Ξ is a countable union of Lipschitz graphs, any point of infinite density in Ξ satisfies Theorem 2.

For Λ^2 we note that the only point of density is $(0,0)$. Note that restricted to $[-1,1]$ the functions making up Λ^2, ($u_n = x^2/n$) are Lipschitz. Thus taking $\rho_1 = 1/2$ and $y = (0,0)$ in Theorem 2 the conditions of Theorem 2 are satisfied so that Λ^2 does not satisfy property (iv). $\diamondsuit$

Remark
We note that in Lemma 6 and Theorem 2 we only used $\delta < 1/8$. Thus the full power of property (iv) has not been used. It is therefore possible and in fact likely that we could force any potential counter examples to (iv) (2) to be even stranger than what we have forced here. Even without using the δ-fine property I believe that an improvement to Theorem 2 could be made in the form of the following conjecture.

Conjecture 4.1.
Suppose $A \subset \mathbb{R}^2$ and that there exists a $y \in A$ such that

$$\mathcal{H}^1(y \cap B_\rho(A)) = \infty \text{ for all } \rho > 0$$

and for some $\rho_1 > 0$,

$$y \in G_y^{-1}(graphu) \cap B_{\rho_1}(y) \text{ and}$$

$$\overline{A \cap G_y^{-1}(graphu)} = G_y^{-1}(graphu) \subset A$$

where $u \in C^0(\mathbb{R};\mathbb{R})$, $G_y \in G(1,2)$ and $G_y(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the rotation such that $G_y(G_y) = \mathbb{R}$.

Then A does not have property (iv).

The idea being that although in this case the full infinite measure could all be produced from the one function, in the case where all the measure does come from the single function it must fold on itself sufficiently tightly and densely to either create a maximum or minimum somewhere we we could apply Lemma 6, or where essentially parallel lines would appear in which case choosing the correct size ball would mean that the approximating affine space would be essentially one of the lines and the intersection with the neighbouring line would then provide a contradiction to A having property (iv).

More quantitatively, we note that there are several methods of attacking the proof and "almost getting there". One method, using Lemma 6, reduces the proof to the following.

Conjecture 4.2.
Suppose I_1, I_2 are compact subintervals of $\mathbb{R}$ and

$$u : I_1 \to I_2.$$

Suppose further that for all $x_1, x_2 \in I_1$ such that $u(x_1) = u(x_2)$

$$\sup\{|u(y) - u(x_1)| : y \in [x_1, x_2]\} < |x_1 - x_2|$$

Then, for any $\delta > 0$ there exists a partition $P = \{p_1, ..., p_Q\}$ of I_1 with

$$\max\{|p_i - p_{i-1}| : 2 \leq i \leq Q\} < \delta\}$$

and

$$\sum_{i=2}^{Q} |u(p_i) - u(p_{i-1})| < C < \infty.$$

Having discussed the non-simplicity of counter examples to (iv) (2), we reform what we have shown in how it is written to emphasise that a set thus need only be (iv) and posses at every point of infinite density a piece of graph to be sure that we have locally finite measure. This is an improvement on previous theory since such sets need not even be weak locally rectifiable. Clearly, we must first give a formal definition of these types of sets.

Definition 4.3.
Let μ be a measure on $\mathbb{R}^{n+k}$. Then $A \subset \mathbb{R}^{n+k}$ is said to posses a piece of Lipschitz graph at $x \in A$ if there exists an $r > 0$, $G \in G(n, n+k)$ and a Lipschitz function $u : G \mapsto G^{\perp}$ such that

$$x \in graphu$$

and

$$\mathcal{H}^n((graphu \sim A) \cap B_r(x)) = 0.$$

Definition 4.4.
A set A is called weak locally countably n-rectifiable if for all $x \in A$ there exists $r > 0$ such that

$$A \cap B_r(x)$$

is countably n-rectifiable.

It is clear that Definition 4.3 is the same condition as that given in Theorem 4.1 so that the claim that this condition together with (iv) leads to locally finite measure follows from the same theorem.

The claim that this is a lesser task to showing rectifiability follows from the existence of n-unrectifiable sets of $\mathcal{H}^n$ finite measure for any n. Thus any set satisfying Definition 4.3 in union with any n-unrectifiable set continues to satisfy the conditions of Definition 4.3.

Chapter 5

Fitting the Counter Examples

We mentioned in Chapter 2 that only questions with the answer "no" remain to be shown. In this section we show these results by appropriately fitting counter examples. For us this means showing firstly that the set in fact satisfies the definition that we claim it does and secondly that the set either has the wrong dimension (i.e. dimension greater than 1) or does not have locally finite $\mathcal{H}^1$-measure depending on which property it is to which we wish to answer "no". As mentioned in the introduction, the higher dimensional cases will be discussed the following chapter. The reason the general dimension is not dealt with here is that they in any case reduce to the 1-dimensional case as we shall see.

There is in fact, in terms of classifying the properties of our defintions, little that remains to be shown. What remains, however, is technical and non-trivial.

Fitting counter the counter example to (iv) (2) in particular shows that a non-rectifiable set (we show that A_ε and $\mathcal{A}_\varepsilon$ are non-recitifiable later) spiralling at all points and magnifications does not spiral too tightly around any given point.

The structure of the Chapter is that we show that Λ_δ satisfies (vi) which will answer (vi) (2) in the negative. We do the same with Λ^2 for (iii). $\mathcal{A}_\varepsilon$ is then shown to satisfy (iv) (actually via first showing that A_ε satisfies (iv)), from which (iv) (2) is answered in the negative, and as a corollary therefore (iii) (2) is also answered in the negative. Finally Γ_ε is shown to satisfy (v), from which it follows that (v) (1) is answered with a no, and therefore as a corollary, the remaining questions: (v) (2), (ii) (1) and (ii) (2) are also answered with no.

The proofs that the sets satisfy the definitions are mainly geometric and will actually mostly involve fitting sets in cones and then considering an appropriate neighbourhood of the center point. For this we need to develop notation to describe the cones we are using. As we will also find sets that should be covered by a cone meeting at a point, notation and theory also need to be developed for angles between sets. The appropriate definitions will be made as (or shortly before) they are used.

Definition 5.1.
Let A be a 1-dimensional affine subspace of $\mathbb{R}^2$, $\delta > 0$ and $x \in \mathbb{R}^n$, then A is said to be a subset of the **δ-cone** *at x, $C_\delta(x)$, if*

$$A \subset \left\{ y = (y_1, y_2) \in \mathbb{R}^n : \frac{y_2}{y_1} < \delta \right\} + x =: C_\delta(x).$$

More generally, if L is a 1-dimensional affine space in $\mathbb{R}^2$, $x \in A \cap L$ and ϕ is the orthogonal transformation such that

$$\phi(L) = \mathbb{R}$$

and

$$\phi(x) = 0$$

then we say that A is a subset of the **δ-cone around L at** *x, $C_{\delta,L}(x)$ if*

$$A \subset \phi^{-1}\left(\left\{ y = (y_1, y_2) \in \mathbb{R}^n : \frac{y_2}{y_1} < \delta \right\} \right) =: C_{\delta,L}(x).$$

5.1 Simple Counter Examples

We now present the relevant classification results following from the simpler counter examples.

Proposition 5.1.
Λ_δ satisfies (vi), and further does not have weak locally finite $\mathcal{H}^1$ measure so that the answer to (vi) (2) (weakly locally finite measure) is no.

Proof:
There are two types of points to consider. If $x \neq (0, 0)$, then if $x = (x_1, x_2)$

$$x \in \text{ graph} \left(\frac{sgn(x_1)sgn(x_2)\delta x}{n} \right)$$

for some $n \in \mathbb{N}$. Then for

$$r_x = \frac{|x|\delta}{4(n+1)},$$

$$\begin{aligned} B_{r_x}(x) \cap \Lambda_\delta &\subset \text{graph}\left(\frac{sgn(x_1)sgn(x_2)\delta x}{n}\right) \\ &\subset G^{\delta r}_{\delta/n,x}, \end{aligned}$$

where $G_{\delta/n} \in G(1,2)$ is the affine space defined by graph$((sgn(x_1)sgn(x_2)\delta x)/n)$, for each $r \in (0, r_x]$. Thus, by setting $L_x = G_{\delta/n,x}$, x is an acceptable point with respect to (vi).

If $x = (0,0)$, then by construction, we may choose $L_x = \mathbb{R}$ and note that

$$G_{\delta/n,x} \subset C_\delta(x)$$

for each $n \in \mathbb{N}$, so that

$$\Lambda_\delta \subset C_\delta(x).$$

It follows that

$$\Lambda_\delta \subset \mathbb{R}^{\delta\rho} = L_x^{\delta\rho}$$

for each $\rho > 0$. Thus choosing a $r_x > 0$ at random we have

$$\Lambda_\delta \subset L_x^{\delta r}$$

for each $r \in (0, r_x]$.

It follows that Λ_δ satisfies (vi).

Note, however, that due to the fact that there are countably infinitly many lines of length $2r$ going through any ball of radius r around $(0,0)$, it follows that for all $r > 0$

$$\mathcal{H}^1(\Lambda_\delta \cap B_r((0,0))) = \infty$$

so that Λ_δ is not weak locally $\mathcal{H}^1$ finite. It follows that the answer to (vi) (2) is no. $\diamondsuit$

Proposition 5.2.
Λ^2 satisfies (iii), and further does not have weak locally finite $\mathcal{H}^1$ measure so that the answer to (iii) is no.

Proof:
There are two types of points to consider. If $x \neq (0,0)$, then if $x = (x_1, x_2)$

$$x \in \text{ graph}\left(\frac{sgn(x_1)sgn(x_2)\delta x^2}{n}\right)$$

for some $n \in \mathbb{N}$. Then for

$$r_x = \frac{|x|^2\delta}{4(n+1)},$$

$$B_{r_x}(x) \cap \Lambda^2 \subset \text{ graph}\left(\frac{sgn(x_1)sgn(x_2)\delta x^2}{n}\right)$$

Since also x^2 is differentiable there is a tangent line L_x to $sgn(x_1)sgn(x_2)x^2/n$ at x and a radius that can be chosen to be smaller than r_x, $r_{x_1} = r_{x_1}(\delta) > 0$, such that for all

$$y \in \text{ graph}\frac{sgn(x_1)sgn(x_2)x^2}{n} \cap B_{r_{x_1}}(x)$$

$$|\pi_{L_x^\perp}(y) - \pi_{L_x^\perp}(x)| < \delta|\pi_{L_x}(y) - \pi_{L_x}(x)|$$

so that

$$B_r(x) \cap \Lambda^2 \subset L_x^{\delta r}$$

for each $r \in (0, r_{x_1}]$. Thus x is an acceptable point with respect to (vi).

If $x = (0,0)$, then by construction, we may choose $L_x = \mathbb{R}$ and note that for $|x| < \delta$

$$\begin{aligned}\frac{|x^2|}{n} &= \frac{|x||x|}{n} \\ &< |x|\delta\end{aligned}$$

for each $n \in \mathbb{N}$. Thus it follows that for each $r \in (0, r_x = \delta]$

$$\Lambda^2 \cap B_r((0,0)) \subset L_x^{r\delta}.$$

It follows that Λ^2 satisfies (vi).

Note, however, that due to the fact that there are countably infinitly many lines of length greater than or equal to $2r$ going through any ball of radius r around $(0,0)$, it follows that for all $r > 0$

$$\mathcal{H}^1(\Lambda^2 \cap B_r((0,0))) = \infty$$

so that Λ^2 is not weak locally $\mathcal{H}^1$ finite. It follows that the answer to (vi) (2) is no. $\diamondsuit$

5.2 Spiralling

For A_ε and $\mathcal{A}_\varepsilon$ we show that the required measure properties hold first. That is that both of the sets are not weak locally $\mathcal{H}^1$-finite. After that we then demonstrate that the set indeed satisfies (iv). Indeed, we have to work quite hard to get the results for Γ_ε and A_ε. This arises from the fact, as has been mentioned and as will be shown in the next chapter, that Γ_ε and A_ε develope spirals in the set. In order to show the required properties we need to show that these spirals are not too tight. We now prove a technical lemma showing that we can find a "spiral free" view of our sets Γ_ε and A_ε. We can then discuss the measure properties of A_ε and $\mathcal{A}_\varepsilon$.

In order to discuss spiralling, we clearly need to discuss angles. For us, most essential will be the angle between two sets, particularly the angle between two trianglular caps. As simply saying the angle between two sets is unclear, we make a definition that will be sufficient for our needs.

Definition 5.2.
Let A and B be two sets with a single common point z that can be divided by some $G \in G(1,2)$ in a sense that is explained below. Then the angle between the two sets ψ_B^A is defined by

$$\psi_B^A = \min\{\theta : C_\theta(z) \supset G(A \cup B) \textit{ for some } G \in G(1,2) \textit{ dividing } A \textit{ and } B\}$$

where as usual $G(1,2)$ is the grassman manifold, $G(\cdot)$ denotes the rotation that takes $G \in G(1,2)$ to $\mathbb{R}_x$, and G divides A and B if for all $X \in A$, $\pi_x(X) \leq 0$ and for all $Y \in B$, $\pi_x(Y) \geq 0$

Remarks: Clearly if $A_1 \subset A$, and $B_1 \subset B$ are such that $A_1 \cap B_1 = A \cap B = \{z\}$ then $\psi_{B_1}^{A_1} \leq \psi_B^A$. Note that the order is important due to the dividing of A and B. The notation ψ_B^A will always denote that A is in the "left cone half" (i.e. $\pi_x(G(A)) \subset \mathbb{R}_x^-$) and B is in the "right cone half" (i.e. $\pi_x(G(B)) \subset \mathbb{R}_x^+$) for the G giving the minimum. We note that $\psi_{(\cdot)}^{(\cdot)}$ is subadditive in the sense that, if A, B and C are sets for which the definition makes sense for the pairings $\{A, B\}$ and $\{B, C\}$ with $z_1 = A \cap B$ and $z_2 \in B \cap C$, then

$$\psi_{C-\{z_2-z_1\}}^A \leq \psi_B^A + \psi_C^B,$$

provided that such a value is less than $\pi/2$ (to ensure the dividing of the sets continues to make sense). Note that $\psi_{(\cdot)}^{(\cdot)}$ is translation and rotation invariant. We note also particularly that in considering the angle between sets A and B, if there is an affine space L such that $A \cap L = \{z, z_a\}$ (i.e. contains the

point common with B, z, and another point), then $\psi_B^L \le \psi_B^A$ otherwise it would be impossible to contain z_a and B in a cone of angle ψ_B^A around z.

We also need to consider the angles that are actually intrinsic to the triangular caps.

Definition 5.3.
Let $n \in \mathbb{N} \cup \{0\}$ and $j \in \{1, 2, ..., 2^n\}$, then we see from Constructions 1,2 and 3 that the triangular cap $T_{n,j}$ is an isosceles triangle. We denote the angles of $T_{n,j}$ as $\theta_{n,j}^A$ and $\pi - 2\theta_{n,j}^A$ where

$$\theta_{n,j}^A = \tan^{-1}\left(\frac{2^{2-n}\varepsilon}{\frac{(1+n16\varepsilon^2)^{1/2}}{2^{n+1}}}\right)$$

and where the ε is that associated with the construction of A_ε. Should the set A be understood we will simply write $\theta_{n,j}$. Further, as in this chapter, should the $\theta_(n,j)$ be independent of j for the understood set A; $\theta_(n,j)$ will be written $\theta_(n,\cdot)$.

Also, suppose that L is an 1-dimensional affine subspace (i.e. a line) of $\mathbb{R}^2$ of finite length (so that it has a middle point l), then we use O_L to denote the orthogonal transformation such that

$$O_L : L \to \mathbb{R}$$

and

$$O_L(l) = (0,0).$$

Remark: At the present time the angles $\theta_{n,j}^A$ are independent of the index j. However, in Chapters 7 and 8 when we look at general forms of the construction of A_ε, the angles will be allowed to vary dependent on n and j. For uniformity and simplicity later in the work, we introduce the symbol for the more general needs immediately.

Note: We note that from here on we take $\psi(0,\varepsilon) < \pi/32$. Thus we need ε such that

$$tan^{-1}\left(\frac{8\varepsilon}{(1+16\varepsilon^2)^{1/2}}\right) < \frac{\pi}{32}$$

(coming from the definition of $\psi(n,\varepsilon)$.) That is

$$\frac{8\varepsilon}{(1+16\varepsilon^2)^{1/2}} < 0.09$$

so that taking

$$0 < \varepsilon < \frac{1}{100}$$

is sufficient. Since we in any case want to look at very small ε and eventually will also be looking at $\varepsilon \to 0$, this presents us with no problems. We will therefore henceforth assume the ε used to construct Γ_ε, A_ε, $\mathcal{A}_\varepsilon$ and other similar sets is less than 0.01. The reason for this assumption is that it is required for the spiralling Lemmas to work.

Lemma 5.1.
Suppose that A_ε, $\mathcal{A}_\varepsilon$ and Γ_ε are as defined in Constructions 1,2 and 3. Then

(1)
should two neighbouring triangles, $T_{n,i}$ and $T_{n,i+1}$, be contained in another (necessarily earlier) triangular cap $T_{m,j(i)}$ $(m \leq n)$ then

$$\psi^{T_n,i}_{T_{n,i+1}} \leq 2\theta_{m,j(i)} \leq 2\theta_{0,1}.$$

and

(2)
the rectangle

$$R_{n,i} = \pi_x \left(O_{A_{n,i}} \left(\cup_{j:|i-j|\leq 1} A_{n,j} \right) \right) \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]$$

has the property

$$O^{-1}_{A_{n,i}}(R_{n,i}) \cap A \subset \bigcup_{j:|i-j|\leq 1} A_{n,j},$$

in the case of A_ε and $\mathcal{A}_\varepsilon$ and

$$O^{-1}_{A_{n,i}}(R_{n,i}) \cap A \subset \bigcup_{j:|i-j|\leq 1} T_{n,j},$$

in the case of Γ_ε

Proof:
We write the proof for A_ε, from which the proofs for $\mathcal{A}_\varepsilon$ and Γ_ε follows. This is true for $\mathcal{A}_\varepsilon$ since $\mathcal{A}_\varepsilon \subset A_\varepsilon$ and it is true for Γ_ε since we make all claims with respect to the triangular caps, and the second claim for A_ε follows by noting that in $\theta_{n,j}$ only $A_{n,j}$ is in A in any case. The only additional tool used is properties of $\theta_{n,j}$. However since the only property of $\theta_{n,j}$ from the construction of A_ε that is used is that $\theta_{n,j} \leq \theta_{m,i}$ for $m \leq n$ and since $\theta_{n,j} \equiv \theta_{0,1}$ for

all $n \in \mathbb{N}$, $j \in \{1, ..., 2^n\}$ in the construction of Γ_ε, all arguments involving $\theta_{\cdot,\cdot}$ also translate directly to Γ_ε.

For (1), let $T_{n,i}$ and $T_{n,i+1}$ be two neighbouring triangular caps with common point z. Then, by the construction of A_ε, $z = z_{n_1+1,2i_1}$ is the vertex of a triangular cap T_{n_1,i_1} for some $n_1 < n$ and some appropriate i_1. Further, since $z \in T_{m,j(i)}$ and $T_{n,i}, T_{n,i+1} \subset T_{m,j(i)}$ so that $z \notin E(A_{m,j(i)})$ $m < n_1$ as otherwise the vertex a_{n_1,i_1} cannot be in $T_{m,j(i)}$.

Then by considering $G_{n_1,i_1} \in G(1,2)$ chosen such that $G_{n_1,i_1} || A_{n_1,i_1}$ we see that we can choose two "halves" (divided at $z_{n_1+1,2i_1}$) of G_{n_1,i_1}, $G^-_{n_1,i_1}$ and $G^+_{n_1,i_1}$, such that

$$\psi^{A_{n_1+1,2i_1}}_{G_{n_1,i_1^+}+z} \leq \theta_{n_1,\cdot} \text{ and } \psi^{G_{n_1,i_1^-}+z}_{A_{n_1+1,2i_1-1}} \leq \theta_{n_1,\cdot}$$

so that, since in both cases in finding the minimum over cones, from which the definition of $\psi^{A_{n_1+1,2i_1}}_{G_{n_1,i_1^+}+z}$ and $\psi^{G_{n_1,i_1^-}+z}_{A_{n_1+1,2i_1-1}}$ comes, we used the cone with respect to $G_{n_1,i-1}$, we have

$$\psi^{A_{n_1+1,2i_1}}_{A_{n_1+1,2i_1-1}} \leq \theta_{n_1,\cdot}.$$

Since then $T_{n_1+1,2i_1-1}$ and $T_{n_1+1,2i_1}$ are constructed on the interior of T_{n_1,i_1} with a base angle of $\theta_{n_1+1,\cdot}$, it follows similarly that

$$\psi^{T_{n_1+1,2i_1}}_{G_{n_1,i_1^+}+z} \leq \theta_{n_1,\cdot} + \theta_{n_1+1,\cdot} \text{ and } \psi^{G_{n_1,i_1^-}+z}_{T_{n_1+1,2i_1-1}} \leq \theta_{n_1,\cdot} + \theta_{n_1+1,\cdot}$$

so that, since we have, as above, in both cases again made the statements about $\psi^{\cdot}_{\cdot}$ with respect to a cone around G_{n_1,i_1}

$$\psi^{T_{n_1+1,2i_1-1}}_{T_{n_1+1,2i_1}} \leq \theta_{n_1,\cdot} + \theta_{n_1+1,\cdot}.$$

Now, since $\theta_{n,\cdot} > \theta_{m,\cdot}$ for all $n < m$ it follows that $\theta_{n_1,\cdot} \leq \theta_{m,\cdot} \leq \theta_{0,\cdot}$ and that $\theta_{n_1+1,\cdot} \leq \theta_{m,\cdot} \leq \theta_{0,\cdot}$ so that

$$\psi^{T_{n_1+1,2i_1-1}}_{T_{n_1+1,2i_1}} \leq 2\theta_{m,\cdot} \leq 2\theta_{0,\cdot}.$$

Finally, we note that now, by construction (in that A_ε is defined through intersection of the constructing levels) that $T_{n,i} \subset T_{n_1,i_1}$ and $T_{n,i+1} \subset T_{n_1,i_1+1}$ so that

$$\psi^{T_{n,i}}_{T_{n,i+1}} \leq 2\theta_{m,\cdot} \leq 2\theta_{0,\cdot}.$$

This proves (1).

For (2), note that since $\varepsilon < 1/100$, $\theta_{0,\cdot} < \pi/32$.

We first need to make a subclaim.

The claim is that if $T_{n,i}$ and $T_{n,j}$ are triangular caps with $2 \leq |i-j| \leq 3$ then

$$\pi_x\left(O_{A_{n,i}}\left(\bigcup_{j:|i-j|<2} T_{n,j}\right)\right) \cap \pi_x(O_{A_{n,i}}(T_{n,j}) - \{z_{n,i-2}, z_{n,i+1}\}) = \emptyset$$

From this claim we will prove (2). As claimed above, we note that since

$$\bigcup_{j:|i-j<2} A_{n,j} = A \cap \bigcup_{j:|i-j|<2} T_{n,j}$$

it is sufficient to prove that for any $T_{n,i}, T_{n,i+1}, T_{n,i+2}$ we have

$$A \cap \pi_x\left(O_{A_{n,i+1}}\left(\bigcup_{j:|i+1-j|\leq 1} T_{n,j}\right)\right) \times [-2\mathcal{H}^1(A_{n,\cdot}), 2\mathcal{H}^1(A_{n,\cdot})] \subset \bigcup_{j:|i+1-j|\leq 1} T_{n,j}.$$

We now consider our claim.

We prove the case for $j - i > 0$, the other case following symmetrically. Note that we know from (1) that

$$\psi_{T_{n,i+1}}^{T_{n,i}} \leq 2\theta_{0,\cdot}$$

and that

$$\psi_{T_{n,i+2}}^{T_{n,i+1}} \leq 2\theta_{0,\cdot}$$

so that

$$\psi_{T_{n,i+2}-(z_{n,i+1}-z_{n,i})}^{T_{n,i}} \leq 4\theta_{0,\cdot}.$$

Indeed, since

$$\psi_{T_{n,i+3}}^{T_{n,i+2}} \leq 2\theta_{0,\cdot},$$

$$\begin{aligned}
\psi_{T_{n,i+2}-(z_{n,i+2}-z_{n,i})}^{T_{n,i}} &= \psi_{T_{n,i+3}-(z_{n,i+2}-z_{n,i+1})-(z_{n,i+1}-z_{n,i})}^{T_{n,i}} \\
&\leq \psi_{T_{n,i+1}}^{T_{n,i}} + \psi_{T_{n,i+2}-(z_{n,i+2}-z_{n,i+1})}^{T_{n,i+1}} \\
&\leq \psi_{T_{n,i+1}}^{T_{n,i}} + \psi_{T_{n,i+2}}^{T_{n,i+1}} + \psi_{T_{n,i+3}}^{T_{n,i+2}} \\
&\leq 6\theta_{0,\cdot}.
\end{aligned}$$

It thus follows that $\psi^{A_{n,i}}_{T_{n,i+3}-(z_{n,i+2}-z_{n,i})} \leq 6\theta_{0,\cdot}$.

Since $A_{n,i}$ is a line meeting the center of the cone

$$C_{6\theta_{0,\cdot}}(G(z_{n,i})) \supset G(A_{n,i} \cup (T_{n,i+3} - (z_{n,i+2} - z_{n,i})))$$

it follows that

$$O_{A_{n,i}}(G^{-1}(C_{6\theta_{0,\cdot}}(G(z_{n,i})))) \subset C_{12\theta_{0,\cdot}}((0, \mathcal{H}^1(A_{n,i})/2))$$

and thus that

$$O_{A_{n,i}}(T_{n,i+3} - (z_{n,i+2} - z_{n,i})) \subset C^+_{12\theta_{0,\cdot}}((0, \mathcal{H}^1(A_{n,i})/2))$$

(where C^+ denotes the RHS of the cone), and therefore from translation invariance of the cone containing a set

$$O_{A_{n,i}}(T_{n,i+3}) \subset C^+_{12\theta_{0,\cdot},\mathbb{R}_x+z_{n,i+2}}(z_{n,i+2}).$$

This being the worse of the two possible j cases ($j = i+1$ and $j = i+2$), an identical proceedure can be used to show that

$$O_{A_{n,i}}(T_{n,i+2}) \subset C^+_{8\theta_{0,\cdot},\mathbb{R}_x+z_{n,i+1}}(z_{n,i+1}).$$

We note that

$$8\theta_{0,\cdot} < 12\theta_{0,\cdot} < \frac{12\pi}{32} < \frac{\pi}{2}.$$

Thus

$$\pi_x(O_{A_{n,i}}(T_{n,i+2} \cup T_{n,i+3})) \subset [\pi_x(O_{A_{n,i}}(z_{n,i+1})), \infty)$$

and

$$\pi_x(O_{A_{n,i}}(T_{n,i+2} \cup T_{n,i+3}) - \{z_{n,i+1}, z_{n,i-2}\}) \subset (\pi_x(O_{A_{n,i}}(z_{n,i+1})), \infty).$$

We find that a similar argument to the above produces

$$O_{A_{n,i}}(T_{n,i+1}) \subset C^+_{4\theta_{0,\cdot},\mathbb{R}_x+z_{n,i}}(z_{n,i}).$$

so that since $4\theta_{0,\cdot} < \pi/2 - \theta_{0,\cdot}$

$$\begin{aligned}
\max\{\pi_x(y) : y \in O_{A_{n,i}}(T_{n,i+1})\} &= \pi_x(O_{A_{n,i}}(z_{n,i+1})) \\
&> \pi_x(O_{A_{n,i}}(z_{n,i})) \\
&= \max\{\pi_x(y) : y \in O_{A_{n,i}}(T_{n,i})\} \\
&= \pi_x(O_{A_{n,i}}(z_{n,i-1})) + \mathcal{H}^1(A_{n,\cdot}) \\
&\geq \max\{\pi_x(y) : y \in O_{A_{n,i}}(T_{n,i-1})\}.
\end{aligned}$$

Thus clearly

$$\pi_x \left(\bigcup_{j:|i-j|<2} O_{A_{n,i}}(T_{n,j}) \right) \subset (-\infty, \pi_x(O_{A_{2,1}}(z_{2,3})],$$

so that

$$\pi_x \left(\bigcup_{j:|i-j|<2} O_{A_{n,i}}(T_{n,j}) \right) \cap \pi_x(O_{A_{n,i}}(T_{n,i+2} \cup T_{n,i+3}) - \{z_{n,i+1}, z_{n,i-2}\}) = \emptyset$$

proving the claim.

We now prove (2) by induction. We first note that for A_0 and A_1 it is obvious, as there are 1 and 2 triangular caps respectively, meaning that A is clearly a subset of any "triple" (using " " as it is actually impossible to choose a triple) of the form required. For A_2 there are four triangular caps, so that there is something to prove. However, we note that for any chosen i every triangle is either in the "triple" around i or has an index j such that $2 \leq |i-j| \leq 3$. Since A is a subset of the four triangles, the required result follows directly from the above proved claim.

We now prove the inductive step. So we suppose that the inductive hypothesis (i.e. (2)) holds for all triples $\{T_{p,i-1}, T_{p,i}, T_{p,i+1}\}$ for a given $p \in \mathbb{N}$ and show that it holds for an arbitrary triple $\{T_{p+1,i-1}, T_{p+1,i}, T_{p+1,i+1}\}$. We set

$$\mathcal{T} = \cup\{T_{p+1,i-1}, T_{p+1,i}, T_{p+1,i+1}\}.$$

Note first that

$$\bigcup_{j:|i-j|<2} T_{p+1,j} \subset \bigcup_{j:|i_1-j|<2} T_{p,j}$$

where $i_1 = (i/2)^{\sqcap} - 1$ ($x^{\sqcap}$ is the smallest integer $q \geq x$), so that the triple is in fact a subset of a triple in the pth construction level. This triple in the pth construction level, by the induction hypothesis contains exactly 6 trianglular caps in the $(p+1)$th construction level, namely $\{T_{p+1}, j\}_{j=2i_1-3}^{2i_1+2}$ with $T_{p+1,i} \in \{T_{p+1,2i_1-1}, T_{p+1,2i_1}\}$. We also have by the inductive hypothesis that

$$A \cap R_{p,i_1} \subset \bigcup_{j=2i_1-3}^{2i_1+2} T_{p+1,j}.$$

It follows that

$$A \cap R_{p+1,i} \cap R_{p,i_1} \subset \bigcup_{j=2i_1-3}^{2i_1+2} T_{p+1,j}.$$

Now, since $i \in \{2i_1 - 1, 2i_1\}$ we see that for all $j \in \{2i_1 - 3, ..., 2i_1 + 2\}$, either $|i - j| < 2$ or $2 \leq |i - j| \leq 3$. From the above proven claim it follows that for each j such that $2 \leq |i - j| \leq 3$, $(T_{p+1,j} \sim \mathcal{T}) \cap R_{p+1,i} = \emptyset$. Thus

$$A \cap R_{p+1,i} \cap R_{p,i_1} \subset \mathcal{T}.$$

The induction then follows in the case that $R_{p+1,i} \subset R_{p,i_1}$, as in this case

$$A \cap R_{p+1,i} = A \cap R_{p+1,i} \cap R_{p,i_1} \subset \mathcal{T}.$$

We therefore prove that this is the case. It is clearly sufficient to show that

$$O_{A_{p,i_1}}(R_{p+1,i}) \subset O_{A_{p,i_1}}(R_{p,i_1})$$

as in this case

$$\begin{aligned} R_{p+1,i} &= O_{A_{p,i_1}} \circ O^{-1}_{A_{p,i_1}}(R_{p+1,i}) \\ &\subset O_{A_{p,i_1}} \circ O^{-1}_{A_{p,i_1}}(R_{p,i_1}) \\ &= R_{p,i_1}, \end{aligned}$$

which is what we need.

Without loss of generality we may assume that

$$\begin{aligned} O_{A_{p,i_1}}(T_{p+1,i}) &\subset \triangle((0,0), (-\mathcal{H}^1(A_{p,j}/2, 0), (0, \varepsilon\mathcal{H}^1(A_{p,j}))) \\ &\subset \triangle((0,0), (-\mathcal{H}^1(A_{p,j}/2, 0), (0, \mathcal{H}^1(A_{p,j})/100)) \end{aligned}$$

where $\triangle(a, b, c)$ denotes the triangle in $\mathbb{R}^2$ with vertices a, b and c. The other cases follow with symmetric arguments.

We have

$$\pi_{O_{A_{p,i_1}}}\left(\bigcup_{j:|i-j|<2} O_{A_{p,i_1}}(T_p + 1, j)\right) \subset$$

$$\left\{t\left(-\mathcal{H}^1(A_{p,j}, -\frac{\mathcal{H}^1(A_{p+1,j})}{100}\right) + (1-t)\left(\frac{\mathcal{H}^1(A_{p,j})}{2}, \frac{2\mathcal{H}^1(A_{p+1,j})}{100}\right) : t \in [0,1]\right\};$$

so that

$$O_{A_{p,i_1}}\left(\bigcup_{j:|i-j|<2} O_{A_{p,i_1}}(T_p + 1, j)\right) \subset \{x = y + z\}$$

where

$$y \in \left\{t\left(-\mathcal{H}^1(A_{p,j}, -\frac{\mathcal{H}^1(A_{p+1,j})}{100}\right) + (1-t)\left(\frac{\mathcal{H}^1(A_{p,j})}{2}, \frac{2\mathcal{H}^1(A_{p+1,j})}{100}\right) : t \in [0,1]\right\}$$

and

$$z \in \left\{ 2s\mathcal{H}^1(A_{p+1,j}) \left(\frac{-4}{100}, 2 \right) : s \in [-1,1] \right\}.$$

That is $O_{A_{p,i_1}}(R_{p,i})$ is a subset of the quadrilateral with vertices

$$V_1 := (-1.54\mathcal{H}^1(A_{p,j}), 2\mathcal{H}^1(A_{p+1,j}))$$

$$V_2 := (0.96\mathcal{H}^1(A_{p,j}), 2.04\mathcal{H}^1(A_{p+1,j}))$$

$$V_3 := (1.04\mathcal{H}^1(A_{p,j}), -2\mathcal{H}^1(A_{p+1,j}))$$

and

$$V_4 := (-1.46\mathcal{H}^1(A_{p,j}), -2.04\mathcal{H}^1(A_{p+1,j})).$$

Noting then that, due to the fact that $\theta_{0,\cdot} < \pi/32$ and the general fact that $\psi_{T_{p,j+1}}^{T_{p,j}} < 2\theta_{0,\cdot}$ (from (1)) we get

$$\begin{aligned} \mathcal{H}^1(\pi_x(O_{A_{p,i_1}}(T_{p,j}))) &> \cos\left(\frac{\pi}{8}\right)\mathcal{H}^1(A_{p,j}) \\ &> 0.9\mathcal{H}^1(A_{p,j}) \end{aligned}$$

for all j such that $|j - i_1| < 2$, and since

$$\begin{aligned} \mathcal{H}^1(A_{p+1,j}) &= \frac{1}{2}(1 + 16\varepsilon^2)^{1/2}\mathcal{H}^1(A_{p,k}) \\ &< 0.6\mathcal{H}^1(A_{p,k}) \end{aligned}$$

we have

$$\begin{aligned} R_{p,i_1} &= O_{A_{p,i_1}}(R_{p,i_1}) \\ &\supset [-1.9\mathcal{H}^1(A_{p,j}), 1.9\mathcal{H}^1(A_{p,j})] \times [-2\mathcal{H}^1(A_{p,j}), 2\mathcal{H}^1(A_{p,j})] \\ &\supset [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})] \times [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})]. \end{aligned}$$

Since clearly

$$V_1, V_2, V_3, V_4 \in [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})] \times [-3\mathcal{H}^1(A_{p+1,j}), 3\mathcal{H}^1(A_{p+1,j})]$$

it follows that

$$O_{A_{p,i_1}}(R_{p+1,i}) \subset O_{A_{p,i_1}}(R_{p,i_1})$$

and thus that

$$R_{p+1,i} \subset R_{p,i_1}$$

completing the proof of (2). $\diamondsuit$

5.3 Measure Properties of A_ε and $\mathcal{A}_\varepsilon$

We now show that A_ε and $\mathcal{A}_\varepsilon$ are not weak locally $\mathcal{H}^1$-finite. We start with the simpler: A_ε.

Lemma 5.2.
Let $\varepsilon > 0$ be such that A_ε is well defined. Then A_ε is not weak locally $\mathcal{H}^1$ finite.

Proof:
We note that for each $n_0 \in \mathbb{N}$, since A_ε makes the lines in A_{n_0} less straight, the refinement to A_ε increases the measure of A_{n_0}. That is

$$\mathcal{H}^1\left(\bigcap_{i=1}^{n_0}\bigcup_{n=i}^{\infty} A_n\right) \geq \mathcal{H}^1(A_{n_0}).$$

Also, for arbitrary $n \in \mathbb{N}$, from Lemma 3.2 we have

$$\mathcal{H}^1(A_n) = (1 + n16\varepsilon^2)^{1/2}.$$

So that

$$\begin{aligned} \mathcal{H}^1(A_\varepsilon) &\geq \liminf_{n\to\infty}(1 + n16\varepsilon^2)^{1/2} \\ &= +\infty \end{aligned}$$

Since for each $n \in \mathbb{N}$, $A_\varepsilon \subset \cap_{n=0}^{\infty} \cup_{i=1}^{2^n} T_{n,i}$ and $\lim_{n\to\infty}$ base length $T_{n,i} = 0$ we know that for each $x \in A$, and for each $R > 0$ there exists n and i such that $x \in T_{n,i} \subset B_R(x)$. Thus also $A_{n,i} \subset B_R(x)$.

We now note that by the construction of A_ε we actually have that the further construction of A_ε on $A_{n,i}$ is the same as that for A_ε except that we start with a base length $\mathcal{H}^1(A_{n,i})$ of $\mathcal{H}^1(A_n)/2^n$ instead of 1. That is $A_\varepsilon \cap T_{n,i}$ is a version of A_ε scaled by a factor of $\mathcal{H}^1(A_{n,j})$. Thus

$$\begin{aligned} \mathcal{H}^1(A_\varepsilon \cap B_R(x)) &\geq \mathcal{H}^1(T_{n,i} \cap B_R(x)) \\ &\geq \frac{\mathcal{H}^1(A_n))}{2^n}\mathcal{H}^1(A_\varepsilon) \\ &= +\infty. \end{aligned}$$

Since this is true for each $x \in A_\varepsilon$ and each $R > 0$ the conclusion follows. $\diamondsuit$

This result also leads to the following interesting result. Not only is it interesting in itself, showing that the set A_ε has infinite density in its own dimension everywhere in the set. It is also useful in showing the nonrectifiability of A_ε later on.

Corollary 5.1.
For each $y \in \overline{A_\varepsilon}$

$$\Theta^1(\mathcal{H}^1, A_\varepsilon, y) = \infty.$$

Proof:
Let $y \in \overline{A_\varepsilon}$ and $\rho > 0$.

Then there is a $y_1 \in A_\varepsilon \cap B_{\rho/2}(y)$ such that $B_{\rho/2}(y_1) \subset B_\rho(y)$.

Since $y_1 \in A_\varepsilon$ for each $n \in \mathbb{N}$ there is a triangular cap $T_{n,i(n,y)} \ni y$. Also, there is an $n_0 \in \mathbb{N}$ such that $\mathcal{H}^1(A_{n,\cdot}) < \rho/4$ for each $n > n_0$ so that $T_{n,i(n,y)} \subset B_{\rho/2}(y_1)$ for each $n > n_0$.

Now From the symmetry of construction we see that $T_{n_0+1,i(n_0+1,y)}$ is a $\mathcal{H}^1(A_{n_0+1,\cdot})$ scale copy of $A_{2^{-n_0}\varepsilon}$. However, from Lemma 5.2 we know $\mathcal{H}^1(A_{2^{-n_0}\varepsilon}) = \infty$, thus

$$\begin{aligned}
\mathcal{H}^1(A_\varepsilon \cap B_\rho(y)) &\geq \mathcal{H}^1(A_\varepsilon \cap B_{\rho/2}(y_1)) \\
&\geq \mathcal{H}^1(A_\varepsilon \cap T_{n_0+1,i(n_0+1,y)}) \\
&= \mathcal{H}^1(A_{n_0+1,\cdot}) \cdot \mathcal{H}^1(A_{2^{-n_0}\varepsilon}) \\
&= \infty.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Theta(A_\varepsilon, y) &= \lim_{\rho \to \infty} \frac{\mathcal{H}^1(B_\rho(y) \cap A_\varepsilon)}{\omega_1 \rho} \\
&= \infty.
\end{aligned}$$

$\diamondsuit$

Although having an infinitely dense point is not that uncommon, and infact having a set of $\mathcal{H}^1$ positive measure of points of $\mathcal{H}^1$ infinite density is not uncommon, that A_ε is a set of positive $\mathcal{H}^1$ measure that has infinite $\mathcal{H}^1$ density at all points of its closure is less common, which makes A_ε a set of peculiar interest in its own right without association to the properties that we are currently discussing.

Although it is possible that$\mathcal{A}_\varepsilon$ has this same peculiar property, it is not easy to prove, and in fact we don't. We settle for finding one such point, however by removing small open balls around such points gauranteed by the proof that follows, we know that there must be at least countably many points in $\mathcal{A}_\varepsilon$ of infinite density.

Although the proof that $\mathcal{A}_\varepsilon$ is not weakly locally $\mathcal{H}^1$ finite is more involved than that for A_ε it is similar. We find approximating sets (subsets of A_n) that we can take a limiting infimum of to bound the measure of $\mathcal{A}_\varepsilon$ from below. We then show that this limiting infimum is infinite. The proof that there is a point of infinite density is then an indirect proof using a covering argument.

Lemma 5.3.
$\mathcal{H}^1(\mathcal{A}_\varepsilon) = \infty$.

Proof:
Let $\delta > 0$ and $\mathcal{B}_\delta$ be a cover of $\mathcal{A}_\varepsilon$ of balls of radius smaller than or equal to δ. Then as $\mathcal{A}_\varepsilon$ is compact we can find a finite subcover

$$\mathcal{B}_f = \{B_{w_i}(x_i)\}_{i=1}^{Q}$$

of balls of radius $w_i < \delta$. Further, since it is a finite collection we can define

$$w_m := \{w_1, ..., w_Q\} > 0.$$

By appropriately selecting 8 balls around each $B_{w_i}(x_i)$ of radius w_i we get a new finite collection of $9Q$ balls with

$$w_m \leq w_i \leq \delta$$

that we relabel

$$\mathcal{B}_f = \{B_{w_i}(x_i)\}_{i=1}^{9Q}$$

such that

$$\mathcal{A}_\varepsilon^{w_m} \subset \bigcup_{i=1}^{9Q} B_{w_i}(x_i) \tag{5.1}$$

and

$$\mathcal{H}^1_\delta(\mathcal{A}_\varepsilon) = \inf\left\{\frac{1}{9}\sum_{i=1}^{9Q} w_i\right\} \tag{5.2}$$

where the infimum is taken over all δ-covers of $\mathcal{A}_\varepsilon$.

Now, suppose that $\gamma > 0$ and that there is an $n_0 \in \mathbb{N}$ such that

$$(A_n - \mathcal{B}) \not\subset (\mathcal{A}_\varepsilon)^\gamma$$

for all $n \geq n_0$.

Then for each $\gamma > 0$ there is a sequence $\{x_n\}$ with $x_n \in A_n \sim \mathcal{B}$ $(m \geq n)$ such that $x_m \notin \mathcal{A}_\varepsilon^\gamma$.

Then, as $\{x_n\}$ is infinite in $[0,1] \times [0, 2\varepsilon]$ there exists a convergent subsequence $\{y_n\}$ where for all $n_0 \in \mathbb{N}$ there is a y_n such that

$$y_n \in A_m \sim \mathcal{B} \text{ for some } m \geq n_0. \tag{5.3}$$

We note that by construction

$$\bigcup_{m=n+2}^{\infty} A_m \subset \bigcup_{i=1}^{2^{n+1}} T_{n+1,i}$$

and the boundaries (of the $T_{n+1,i}$'s) closest to A_n are then the $(A_{n+2,i})$. Since then the angle between an $A_{n,\cdot}$ and an $A_{n+2,\cdot}$ that meet is $\theta_{n,\cdot} - \theta_{n+1,\cdot}$; that for all $B_{r_i}(x_i) \in \mathcal{B}$ such that $A_n \cap B_{r_i}(x_i) \neq \emptyset$ we have

$$i \leq 2 + \sum_{j=0}^{n} 2^j (< \infty)$$

and there exists a

$$\min\left\{r_i : 1 \leq i \leq 2 + \sum_{j=0}^{n} 2^j\right\}.$$

Since also, by Lemma 5.1 (2) the closest $A_{n+2,j}$ to an $A_{n,i} - \cup\{B_r(x) : x \in E(A_{n,i})$ must be an $A_{n+2,j} \subset A_{n,i}$ we then have

$$\begin{aligned}
d_n &:= d\left(A_n - \mathcal{B}, \bigcup_{i=n+1}^{\infty} A_n\right) \\
&\geq d(A_n - \mathcal{B}, A_{n+2} - \mathcal{B}) \\
&\geq sin^{-1}(\theta_{n+1,\cdot} - \theta_{n+2,\cdot}) \cdot \inf\left\{r_i : i \leq 2 + \sum_{j=0}^{n+2} 2^j\right\} \\
&> 0.
\end{aligned} \tag{5.4}$$

Thus from (5.3) and (5.4) we have

$$\lim_{n\to\infty} y_n \in \overline{\left(\bigcup_{n=1}^{\infty} (A_n - \mathcal{B})\right)}$$

and

$$\lim_{n\to\infty} y_n \notin A_m$$

for all $m \in \mathbb{N}$ which means

$$\lim_{n\to\infty} y_n \in \mathcal{A}_\varepsilon$$

but for all $n \in \mathbb{N}$, $y_n \notin \mathcal{A}_\varepsilon^\gamma$ so that $d(\mathcal{A}_\varepsilon, y_n) > \gamma$ for all $n \in \mathbb{N}$ which implies

$$d(\lim_{n\to\infty} y_n, \mathcal{A}_\varepsilon) \geq \gamma.$$

This contradiction means that for all $\gamma > 0$ and all $n_o \in \mathbb{N}$ there is an $n > n_0$ such that

$$(A_n - \mathcal{B}) \subset \mathcal{A}_\varepsilon^\gamma.$$

Thus for each $n_0 \in \mathbb{N}$ there is an $n.n_0$ such that

$$(A_n - \mathcal{B}) \subset \mathcal{A}_\varepsilon^{w_m}$$

so that by (5.1) $\mathcal{B}_f$ is a cover of balls of radius smaller than or equal to δ for $A_n - \mathcal{B}$ and therefore

$$\mathcal{H}^1_\delta(A_n - \mathcal{B}) \leq \sum_{i=1}^{9Q} w_i.$$

Since this is true for some $n \geq n_0$ for any $n_0 \in \mathbb{N}$ it follows that

$$\liminf_{n\to\infty} \mathcal{H}^1_\delta(A_n - \mathcal{B}) \leq \sum_{i=1}^{9Q} w_i.$$

Since this is true for any cover of $\mathcal{A}_\varepsilon$ of balls with radius bounded above by δ it follows that

$$\liminf_{n\to\infty} \mathcal{H}^1_\delta(A_n - \mathcal{B}) \leq \inf\left\{\sum_{i=1}^{9Q} w_i\right\}$$

where the infimum is taken over all δ-covers of $\mathcal{A}_\varepsilon$. Thus by (5.2)

$$\liminf_{n\to\infty} \mathcal{H}^1_\delta(A_n - \mathcal{B}) \leq 9\mathcal{H}^1_\delta(\mathcal{A}_\varepsilon).$$

Define

$$L_n := A_n - \mathcal{B}.$$

We will show that

$$\lim_{n\to\infty} \mathcal{H}^1(L_n) = \infty.$$

Before we do this, however, we show how such a fact can be used to complete the proof.

Assuming $\mathcal{H}^1(L_n) = \infty$ we then know that for all $M \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that
$$\mathcal{H}^1(L_n) > 2M$$
for all $n \geq n_0$. Let $n_0 \in \mathbb{N}$ be such a number and let
$$L_{M,m} := \{n \in \mathbb{N} : n \geq n_0 \text{ and } \mathcal{H}^1_{1/m}(L_n) < M\}.$$
Since $\mathcal{H}^1_\delta(A)$ is increasing as δ decreases for any $A \subset \mathbb{R}^2$, $L_{M,m_1} \subset L_{M,m_2}$ whenever $m_1 > m_2$.

Then suppose there is no $m \in \mathbb{N}$ such that $L_{M,m} = \emptyset$, then
$$\bigcap_{m=1}^{\infty} L_{M,m} \neq \emptyset$$
so that there is an $n \geq n_0$ such that $\mathcal{H}^1_{1/m}(L_n) < M$ for all $m \in \mathbb{N}$. Thus
$$\begin{aligned} \lim_{\delta \to 0} \mathcal{H}^1_\delta(L_n) &= \lim_{m \to \infty} \mathcal{H}^1_{1/m}(L_n) \\ &< M \end{aligned}$$
contradicting $\mathcal{H}^1(L_n) \geq 2M$.

It follows that there is a $\delta(M) > 0$ such that $\mathcal{H}^1_{\delta(M)}(L_n) \geq M$ for all n such that $\mathcal{H}^1(L_n) \geq 2M$. Thus
$$\liminf_{n \to \infty} \mathcal{H}^1_{\delta(M)}(L_n) \geq M.$$
Thus for all $M \in \mathbb{R}$ there is a $\delta(M) > 0$ such that for all $\delta < \delta(M) > 0$
$$\begin{aligned} \mathcal{H}^1_\delta(\mathcal{A}_\varepsilon) &\geq \mathcal{H}^1_{\delta(M)}(\mathcal{A}_\varepsilon) \\ &\geq \frac{1}{9} \liminf_{n \to \infty} \mathcal{H}^1_{\delta(M)}(L_n) \\ &\geq \frac{M}{9}. \end{aligned}$$
Thus
$$\begin{aligned} \mathcal{H}^1(\mathcal{A}_\varepsilon) &= \lim_{\delta \to 0} \mathcal{H}^1_\delta(\mathcal{A}_\varepsilon) \\ &= \infty. \end{aligned}$$
To complete the proof, we therefore now need to show that
$$\lim_{n \to \infty} \mathcal{H}^1(L_n) = \infty.$$

We consider first L_7. Note

$$\begin{aligned}\max\{r_i : B_{r_i}(x_i)(A_7) \neq \emptyset\} &= r_1 \\ &= \frac{1}{4}2^{-7}(1 + 7 \cdot 16\varepsilon^2)^{1/2} \\ &< \frac{1}{2}\mathcal{H}^1(A_{7,\cdot})\end{aligned}$$

and thus from Lemma 5.1 (2) we know that this implies

$$\begin{aligned}A_7 \cap B_{r_i}(x_i) &\subset \cup\{T_{7,j} : x_1 \in E(T_{7,j})\} \\ &= T_{7,1}.\end{aligned}$$

We remove this triangular cap from the measure that we count toward L_7 and note that

$$\begin{aligned}\sum_{B_{r_i}(x_i):B_{r_i}(x_i)\cap A_7 \neq \emptyset} r_i &\leq \sum_{i=2}^{\infty} r_i \\ &\leq \sum_{i=1}^{\infty} r_i \\ &< \sum_{i=1}^{\infty} 4^{-i}2^{-7}(1 + 7 \cdot 16\varepsilon^2)^{1/2} \\ &< 2^{-6}.\end{aligned}$$

Each ball $B_{r_1}(x_i)$ such that $B_{r_i}(x_i) \cap A_7 \neq \emptyset$ (which by construction are those $B_{r_i}(x_i)$ such that $x_i = a_{n_i}$ for $n \leq 7$), again by Lemma 5.1 (2) meets only the two adjacent $A_{7,\cdot}$ so that by letting

$$D_n := \{\text{triangular caps discarded from the estimation of } L_n\}$$

we have $D_7 = \{T_{7,1}\}$ and thus

$$\mathcal{H}^1\left(\left(A_7 \sim \bigcup_{T_{7,i}\in D_7} T_{7,i}\right) \cap \mathcal{B}\right) \leq 2^{-6}$$

and since each triangle $T_{7,i}$ gives the same value from

$$\mathcal{H}^1(A_7 \cap T_{7,i})$$

we have with

$$N_n := \mathrm{card}(D_n)$$

$$\mathcal{H}^1\left(A_7 \sim \bigcup_{T_{7,i}\in D_7} T_{7,i}\right) = \frac{2^7 - N_7}{2^7}\mathcal{H}^1(A_7)$$

and thus

$$\mathcal{H}^1(L_7) \geq \frac{2^7 - N_7}{2^7}\mathcal{H}^1(A_7) - 2^{-6}.$$

We note in particular that $N_7 = 1 < 2$ and we make the following inductive claims.

For each $n \geq 7$, by removing triangular caps intersecting

$$\cup\{B_{r_i}(x_i) : i \leq n-6\}$$

we have $N_n \leq 2^{n-5} - 2$ so that

$$\begin{aligned} M_n &:= \{\text{ triangular caps remaining at the } n\text{th stage}\} \\ &\geq 2^{-n} - (2^{n-5} - 2). \end{aligned}$$

Further, we have that

$$\sup\{r_1 : r > n-6\} < \frac{1}{2}\mathcal{H}^1(A_{n,\cdot})$$

and that

$$\mathcal{H}^1\left(\bigcup_{i>n-6} B_{r_i}(x_i) \cap A_n\right) \leq \sum_{i=1}^{\infty} r_i < 2^{-6}$$

so that

$$\begin{aligned} \mathcal{H}^1(L_n) &> \mathcal{H}^1\left(A_n \sim \bigcup_{T_{n,i}\in D_n} T_{n,i} \sim \left(A_n \cap \bigcup_{i>n-6} B_{r_i}(x_i)\right)\right) \\ &\geq \mathcal{H}^1\left(A_n \sim \bigcup_{T_{n,i}\in D_n} T_{n,i}\right) - 2^{-6} \\ &\geq \frac{2^n - (2^{n-5} - 2)}{2^n}\mathcal{H}^1(A_n) - 2^{-6}. \end{aligned}$$

We know that these conditions hold for $n = 7$. Now we assume that they hold for some $n \geq 7$ and show the inductive step to show that they hold for $n+1$.

First, we know that $N_n \leq 2^{n-5} - 2$. This is the number of triangular caps

that we have removed due to the intersection with balls $\{B_{r_i}(x_i)\}_{i=1}^{n-6}$. Thus there exist no more than the triangular caps $T_{n+1,j}$ such that

$$T_{n+1,j} \subset T_{n,i} \in D_n$$

for some i. For each such $T_{n,i} \in D_n$ there are 2 such triangular caps $T_{n+1,j}$.

Then, as

$$r_{n+1-6} < 4^{-(n+1-6}\mathcal{H}^1(A_{7,\cdot}) < \frac{1}{2}\mathcal{H}^1(A_{n+1-6,\cdot})$$

it follows from Lemma 5.1 (2) that $B_{r_{n+1-6}}(x_{n+1-6})$ intersects at most 2 triangular caps $T_{n+1,i}$. Thus removing these triangular caps means that removal of triangular caps of the $n+1$th level due to intersections with $\{B_{r_i}(x_i)\}_{i=1}^{n+1-6}$ has led to the removal of

$$2N_n + 2 \leq 2(2^{n-5} - 2) + 2 = 2^{(n+1)-5} - 2$$

triangular cpas at the $n+1$th level. It follows that

$$M_{n+1} = 2^{n+1} - N_{n+1} \geq 2^{n+1} - (2^{(n+1)-5} - 2)$$

as required.

Now

$$\begin{aligned} \sup\{r_i : i > n+1-6\} &< \left(\frac{1}{4}\right)^{n+1-6} \mathcal{H}^1(A_{7,\cdot}) \\ &< \frac{1}{2}\mathcal{H}^1(A_{n+1,\cdot}). \end{aligned}$$

Since, by construction

$$B_{r_i}(x_i) \cap A_{n+1} = \emptyset \text{ for each } i > 2 + \sum_{j=0}^{n+1} 2^j$$

and since for each $i \leq 2 + \sum_{j=0}^{n+1} 2^j$,

$$x - i \in E(A_{n+1})$$

we know that apart from

$$\bigcup_{i=1}^{n+1-6} B_{r_i}(x_i),$$

for which we have already removed the relevant triangular caps, $B_{r_i}(x_i)$ is a ball around an edge point of A_{n+1} with $r_i < 1/2\mathcal{H}^1(A_{n+1,\cdot})$. Thus by Lemma 5.1 (2) $B_{r_i}(x_i)$ can only meet the two triangular caps $T_{n+1,i}$ with intersection point x_i, it follows that for

$$n+1-6 \leq i \leq 2+\sum_{j=0}^{n+1} 2^j$$

$B_{r_i}(x_i)$ consists of two straight lines of length r_i. Thus

$$\begin{aligned}
\mathcal{H}^1\left(\bigcup_{i=n+1-6}^{\infty} B_{r_i}(x_i)\cap A_{n+1}\right) &= \mathcal{H}^1\left(\bigcup_{i=n+1-6}^{2+\sum_{j=0}^{n+1}2^j} B_{r_i}(x_i)\cap A_{n+1}\right)\\
&\leq \sum_{i=n+1-6}^{2+\sum_{j=0}^{n+1}2^j} 2r_i\\
&< \sum_{i=n+1-6}^{\infty} 2^{-i}\\
&< 2^{-6}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{H}^1(L_{n+1}) &= \mathcal{H}^1(A_{n+1}\sim\mathcal{B})\\
&\geq \mathcal{H}^1\left(A_{n+1}\sim\bigcup_{T_{n+1,i}\in D_n} T_{n+1,i}\sim\bigcup_{i=n+1-6}^{\infty} B_{r_i}(x_i)\right)\\
&\geq \mathcal{H}^1\left(A_{n+1}\sim\bigcup_{T_{n+1,i}\in D_n} T_{n+1,i}\right)-\mathcal{H}^1\left(A_{n+1}\cap\bigcup_{i=n+1-6}^{\infty} B_{r_i}(x_i)\right)\\
&\geq \mathcal{H}^1\left(A_{n+1}\sim\bigcup_{T_{n+1,i}\in D_n} T_{n+1,i}\right)-2^{-6}.
\end{aligned}$$

Since $\mathcal{H}^1(A_{n+1}\cap T_{n+1,i})$ is constant over i we know

$$\mathcal{H}^1\left(A_{n+1}\sim\bigcup_{T_{n+1,i}\in D_n} T_{n+1,i}\right)\geq\frac{M_{n+1}}{2^{n+1}}\mathcal{H}^1(A_{n+1})$$

so that

$$\mathcal{H}^1(L_{n+1})\geq\frac{2^{n+1}-(2^{n+1-5}-2)}{2^{n+1}}\mathcal{H}^1(A_{n+1})-2^{-6}$$

completing the inductive step.

We thus, most importantly have

$$\begin{aligned}\mathcal{H}^1(L_n) &\geq \frac{2^n - (2^{n-5}-2)}{2^n}\mathcal{H}^1(A_n) - 2^{-6} \\ &\geq (1-2^{-4})\mathcal{H}^1(A_n) - 2^{-6}\end{aligned}$$

for each $n \geq 7$, so that

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{H}^1(L_n) &\geq \liminf_{n\to\infty}(1-2^{-4})\mathcal{H}^1(A_n) - 2^{-6} \\ &= -2^{-6} + (1-2^{-4})\liminf_{n\to\infty} \mathcal{H}^1(A_n) \\ &= \infty.\end{aligned}$$

Thus we can use the word limit and write

$$\lim_{n\to\infty} \mathcal{H}^1(L_n) = \infty$$

which is what was required to complete the proof. $\diamondsuit$

Corollary 5.2.
A_ε and $\mathcal{A}_\varepsilon$ are not weak locally $\mathcal{H}^1$-finite.

Proof:
For A_ε this follows directly from Corollary 5.1.

Now, suppose that $\mathcal{A}_\varepsilon$ is weakly locally $\mathcal{H}^1$-finite. Then for each $y \in \mathcal{A}_\varepsilon$ there is a radius $\rho_y > 0$ such that

$$\mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_{\rho_y}(y)) < \infty.$$

$\{B_{\rho_y}(y)\}_{y\in\mathcal{A}_\varepsilon}$ is an open cover of $\mathcal{A}_\varepsilon$ so that since $\mathcal{A}_\varepsilon$ is compact there must exist a finite subcover $\{B_{\rho_{y_n}}(y_n)\}_{n=1}^{Q}$ of $\mathcal{A}_\varepsilon$ and further we know that

$$M := \max\{\mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_{\rho_{y_n}}(y_n)) : 1 \leq n \leq Q\} < \infty.$$

It follows that

$$\begin{aligned}\mathcal{H}^1(\mathcal{A}_\varepsilon) &\leq \mathcal{H}^1\left(\mathcal{A}_\varepsilon \cap \bigcup_{n=1}^{Q} B_{\rho_{y_n}}(y_n)\right) \\ &\leq \sum_{n=1}^{Q} \mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_{\rho_{y_n}}(y_n)) \\ &\leq QM \\ &< \infty.\end{aligned}$$

This contradiction implies that there must exist a $y \in \mathcal{A}_\varepsilon$ such that for each $\rho > 0$

$$\mathcal{H}^1(\mathcal{A}_\varepsilon \cap B_\rho(y)) = \infty$$

and therefore that $\mathcal{A}_\varepsilon$ is not weak locally $\mathcal{H}^1$-finite. $\diamondsuit$

5.4 Approximate j-Dimensionality of A_ε and $\mathcal{A}_\varepsilon$

Having shown the measure theoretic properties of A_ε and $\mathcal{A}_\varepsilon$ that are required for them to be appropriate counter examples to (iv) (2), we now go on to show that A_ε and $\mathcal{A}_\varepsilon$ actually do satisfy the requirements of the Definition of (iv).

Lemma 5.4.
A_ε and $\mathcal{A}_\varepsilon$ satisfy property (iv).

Proof:
Since $\mathcal{A}_\varepsilon \subset A_\varepsilon$, proving that A_ε satisfies (iv) is sufficient to prove the Lemma. We therefore proceed to prove that A_ε is (iv).

We first consider an arbitrary triangular cap, $T_{n,i}$ from somewhere in our construction. From the construction it is clear that it must be isosceles. From Lemma 5.1 and Construction 3.2 (particularly the constructed vertical heights, and Lemma 5.1 (1)) we see that it must have the two sorts of angles, $\psi(n, \varepsilon)$ and $\pi - 2\theta_{n,\cdot}$, where, as in Definition 5.3

$$\theta_{n,\cdot} = \tan^{-1}\left(\frac{2^{2-n}\varepsilon}{\frac{(1+n16\varepsilon^2)^{1/2}}{2^{n+1}}}\right).$$

so that we have

$$\begin{aligned}\lim_{n\to\infty} \theta_{n,\cdot} &= \lim_{n\to\infty} \tan^{-1}\left(\frac{2^{2-n}\varepsilon}{\frac{(1+n16\varepsilon^2)^{1/2}}{2^{n+1}}}\right) \\ &= \tan^{-1}\left(\frac{2^{2-n+n+1}\varepsilon}{(1+n16\varepsilon^2)^{1/2}}\right) \\ &= 0. \end{aligned} \tag{5.5}$$

We now choose arbitrarily some $\delta > 0$ and $x \in A_\varepsilon$. We show that there is a r_x such that for each $r \in (0, r_x]$

$$A_\varepsilon \cap B_r(x) \subset L^{\delta r}_{x,r}.$$

Since the endpoints of $A_{n,i}$ for each n, i are not in A_ε, x is not an endpoint so that we know from (5.5) that we can choose an $r_x > 0$ such that

$$B_{r_x}(x) \cap A \subset T_{n_0,j_0}$$

for some choice of $n_0 \in \mathbb{N}$ and $j_0 \in \{1, ..., 2^{n_0}\}$ and such that for all $n > n_0$

$$\begin{aligned} \theta_\delta &:= tan^{-1}(\delta) \\ &> 3\theta_{n-1,\cdot} + 2\theta_{n-2,\cdot} \\ &> \theta_{n,\cdot}. \end{aligned}$$

Since $x \in A_\varepsilon \cap T_{n_0,j_0}$, for each $n > n_0$, $x \in T_{n,j(n)}$ for some $j(n) \in \{1, ..., 2^n\}$. For each $r \in (0, r_x]$ we can therefore choose an $n_1 > n_0$ and $j_1 = j(n_1)$ such that

$$\mathcal{H}^1(A_{n_1,j_1}) \in [r/2, r).$$

We now consider x as simply being some element of T_{n_1,j_1} and set $L_{r,x}$ to be the affine space parallel to A_{n_1,j_1} containing x.

We now check that

$$2^{2-n_1}\varepsilon > \frac{\delta r}{2}.$$

First, we note that

$$\begin{aligned} \delta &> tan(\theta_{n_1,\cdot}) \\ &= \frac{2^{2-n_1+n_1+1}\varepsilon}{(1 + n16\varepsilon^2)^{1/2}} \\ &= \frac{8\varepsilon}{(1 + n_1 16\varepsilon^2)^{1/2}} \end{aligned}$$

which we get from the selection of n_1. Also, from the selection of n_1 with respect to r that we have

$$\begin{aligned} r &> \mathcal{H}^1(A_{n_1}, j_1) \\ &= \frac{(1 + (n_1)16\varepsilon^2)^{1/2}}{2^{n_1}} \end{aligned}$$

so that

$$\begin{aligned} \delta r &> \frac{8\varepsilon(1 + (n_1)16\varepsilon^2)^{1/2}}{(1 + n_1 16\varepsilon^2)^{1/2} 2^{n_1}} \\ &\geq 2^{3-n_1} \end{aligned}$$

giving the desired inequality. This gives us that the vertical height of T_{n_1,j_1} is less than half the diameter of the neighbourhood that we need around $L_{r,x}$ (that is $L^{\delta r}_{r,x}$). Thus

$$B_r(x) \cap T_{n_1,j_1} \subset L^{\delta R}_{r,x}.$$

It only remains to show that the remainder of $A_\varepsilon \cap B_r(x)$ is inside of an appropriate cone around $L^{\delta r}_{r,x}$. Since from the choice of n_1 with respect to r we have that

$$B_r(x) \subset \pi_x\left(O_{A_{n,i}}\left(\cup_{j:|i-j|\leq 1} A_{n,j}\right)\right) \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})].$$

Thus from Lemma 5.1 (2) it follows that the remainder of A is contained in

$$\bigcup_{i:0<|i-j|<3} T_{n_1,i}$$

so that it suffices to prove that these four caps are in the appropriate cone around $L^{\delta r}_{r,x}$. We note that the union of these four caps is the subset of three T_{n_1-1}, k caps,

$$T_{n_1,j_1-2} \cap T_{n_1,j_1-1} \cap T_{n_1,j_1+1} \cap T_{n_1,j_1+2} \subset T_{n_1-1,j_1-1} \cap T_{n_1-1,j_1} \cap T_{n_1-1,n_1+1}.$$

By construction the maximal angle divergence from $L^{\delta r}_{r,x}$ that an edge on a neighbouring triangular cap of order n_1 is $2\theta_{n-1,\cdot}$ and similarly for a triangular cap of order $n_1 - 1$, the maximal angular divergence is $2\theta_{n-2,\cdot}$. Adding these together (which is actually worse than could possibly occur) we find that the maximal angle *requirement* for a cone around $L^{\delta r}_{r,x}$ is

$$\begin{aligned} 2\theta_{n-1,\cdot} + 2\theta_{n-2,\cdot} &< 3\theta_{n-1,\cdot} + 2\theta_{n-2,\cdot} \\ &< \theta_\delta. \end{aligned}$$

It follows that we now have

$$B_r(x) \cap A \subset L^{\delta r}_{r,x}.$$

Since x and δ were arbitrary, this shows that A_ε has the fine weak 1-dimensional ε-approximation property with local r_x uniformity, (that is, it satisfies (iv)) and thus completes the proof. ♢

Corollary 5.1 and Lemma 5.4 allow us to provide the answer to question (iv) (2). We present this result formally in the following Theorem.

Theorem 5.1.
The answer to (iv) (2) is no.

Proof:
From Lemma 5.4 A_ε is a set that satisfies (iv) (2). Since, from Corollary 5.1 we know that A_ε is not weak locally $\mathcal{H}^1$-finite it follows that A_ε is a counter example to the answer to (iv) (2) being yes. ♢

5.5 Approximate j-Dimensionality of Γ_ε

As previously discussed, the remainder of the answers to our definitions are completely dependent on showing that Γ_ε satisfies (v). We show that this is true, or at least sufficiently true in the following Lemma. Sufficiently true here means that we can find an appropriate ε such that Γ_ε constructed with this ε satisfues (v) for any given $\delta > 0$. This is sufficient since definition (v) is dependent on some arbitrary but fixed δ unlike (iv) which requires δ to be able to be chosen arbitrarily for any set satisfying (iv). We show first that Γ_ε satisfies (v) and then how the remainding classification follows.

Lemma 5.5.
For all $\delta > 0$ there exists an $\varepsilon_\delta = \varepsilon_\delta(\delta) > 0$ such that $\Gamma_{\varepsilon_\delta}$ satisfies property (v) with respect to δ.

Proof:
Let $0 < \varepsilon < 1/100$. We show, in fact, that there exists a function

$$\delta(\varepsilon) : \mathbb{R} \to \mathbb{R}$$

such that

$$\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$$

such that Γ_ε satisfies (v) with respect to $\delta(\varepsilon)$. It then follows that for all $\delta > 0$ there is an $\varepsilon_\delta > 0$ such that $\delta(\varepsilon_\delta) < \delta$; Γ_ε then satisfies (v) with respect to $\delta(\varepsilon_\delta)$ and therefore with respect to δ.

Let $w \in \Gamma_\varepsilon$ and $\rho \in (0, \rho_0](= (0, 1])$. Then, as in Lemma 5.4, we know that there exists an $n \in \mathbb{N}$ such that $w \in T_{n,i}$ for some i with $\mathcal{H}^1(T_{n,i}) \in [\rho, 2\rho)$.

Now from Lemma 5.1 (1)

$$\{\psi_{T_{n,j+1}}^{T_{n,j}}\}_{j=i-1}^{j=i} < 2\theta_{0,\cdot} < \frac{\pi}{16}$$

so that

$$\begin{aligned} O_{A_{n,i}}^{-1}(R_{n,i}) &= O_{A_{n,i}}^{-1}(\pi_x(O_{A_{n,i}}(\cup_{j:|i-j|\le 1} A_{n,j})) \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]) \\ &\supset O_{A_{n,i}}^{-1}([-(0.5\mathcal{H}^1(T_{n,\cdot}) + 0.9\mathcal{H}^1(T_{n,\cdot})), 0.5\mathcal{H}^1(T_{n,\cdot}) + 0.9\mathcal{H}^1(T_{n,\cdot})] \\ &\quad \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]) \\ &\supset O_{A_{n,i}}^{-1}([-\rho, \rho] \times [-2\mathcal{H}^1(A_{n,i}), 2\mathcal{H}^1(A_{n,i})]). \end{aligned}$$

This implies that

$$B_\rho(w) \subset O_{A_{n,i}}^{-1}(R_{n,i}). \tag{5.6}$$

From Lemma 5.1 (2) it follows that

$$\Gamma_\varepsilon \cap B_\rho(w) \subset \bigcup_{j:|i-j|<2} T_{n,j} \cup O^{-1}_{A_{n,i}}(R_{n,i})^c,$$

Since, from (5.6)

$$\begin{aligned} B_\rho(w) \cap O^{-1}_{A_{n,i}}(R_{n,i})^c = \emptyset, \\ \Gamma_\varepsilon \cap B_\rho(w) \subset \bigcup_{j:|i-j|<2} T_{n,j} \end{aligned} \qquad (5.7)$$

and more importantly, that

$$\Gamma_\varepsilon \cap \left(B_{\rho(w)} \sim \bigcup_{j:|i-j|<2} T_{n,j} \right) = \emptyset.$$

Since

$$\begin{aligned} \sup\{\pi_y(x) : x \in O_{A_{n,i}}(T_{n,i})\} &\leq \varepsilon \mathcal{H}^1(A_{n,i}) \\ &\leq \varepsilon 2\rho \end{aligned}$$

and since from Lemma 5.1 (2)

$$O_{A_{n,i}} \left(\bigcup_{j:|i-j|=2} T_{n,j} \right) \subset C_{4\theta_{0,\cdot}}((0,0))$$

so that we have

$$\sup \left\{ |\pi_y(z)| : z \in O_{A_{n,i}} \left(\bigcup_{j:|i-j|=1} T_{n,j} \cap B_\rho(w) \right) \right\} \leq \sin(4\theta_{0,\cdot})\rho$$

it follows that

$$\begin{aligned} \sup\{|\pi_y(z)| : z \in O_{A_{n,i}}(\Gamma_\varepsilon \cap B_\rho(w))\} &< \sup\{2\varepsilon, sin(4\theta_{0,\cdot})\}\rho \\ &= sin(4\theta_{0,\cdot})\rho \end{aligned}$$

and thus by choosing $L_{w,\rho}||A_{n,i}$ we have

$$\sup\{|\pi^\perp_{L_{w,\rho}}(z)| : z \in \Gamma_\varepsilon \cap B_\rho(w)\} < sin(4\theta_{0,\cdot})\rho,$$

that is

$$\Gamma_\varepsilon \cap B_\rho(w) \subset L^{sin(4\theta_{0,\cdot})\rho}_{w,\rho}.$$

Thus Γ_ε satisfies (v) for $\delta > sin(4\theta_{0,\cdot})$. Which, since

$$\lim_{\varepsilon \to 0} sin(4\theta_{0,\cdot}) = 0,$$

by setting $\delta(\varepsilon) = sin(4\theta_{0,\cdot})$, proves the lemma. ♢

The dimension of Γ_ε follows from the work of Hutchinson [10]. The proof is not trivial and so we do not present the proof here. We will however apply Hutchinsons proof regularly as a fundamental theorem of dimension to which we can reduce all of our investigations into the dimension of the generalised Koch Sets considered in Chapters 7 and 8. It is therefore important to state the Theorem and to show that Γ_ε satisfies the conditions required for the Theorem to be applied.

We first mention a result of Mandelbrot [13] required to make sense of the result in [10] that we use.

Proposition 5.3.
Let $\{r_i\}_{i=1}^N$ be a sequence of positive real numbers, then there exists a unique $D \in \mathbb{R}$ such that

$$\sum_{i=1}^{N} r_i^D = 1.$$

With this D we can now consider the appropriate result about dimension from [10].

Theorem 5.2.
If

$$K = \bigcup_{i=1}^{N} S_i(K)$$

where S_i are contraction mappings and if there exists an open set O such that

1. $0 \neq \emptyset$
2. $\bigcup_{i=1}^N S_i(O) \subset O$
3. $S_i(O) \cap S_j(O) = \emptyset$ *whenever* $i \neq j$

Then if $LipS_i =: r_i$ for each $1 \leq i \leq N$ and D is the unique real number for which

$$\sum_{i=1}^{N} r_i^D = 1$$

$$dimK = D.$$

We can apply this Theorem directly to our case with Γ_ε by appealing to Proposition 3 as follows.

Lemma 5.6.
For each $\varepsilon > 0$, $dim\Gamma_\varepsilon > 1$.

Proof:
By Proposition 3.1 there exist, for each $\varepsilon > 0$ contraction maps S_1, S_2 with $\text{Lip}S_i = l(\varepsilon) > 1/2$ for each $i = 1, 2$ and an open set O such that the requirements of Theorem 5.2 are satisfied for $K = \Gamma_\varepsilon$.

It follows that

$$\sum_{i=1}^{2} (\text{Lip}S_i)^{\dim\Gamma_\varepsilon} = 1.$$

That is

$$2l^{\dim\Gamma_\varepsilon} = 1,$$

or

$$\dim\Gamma_\varepsilon = -\frac{\ln 2}{\ln l} > 1.$$

$\diamondsuit$

We now have the tools to, and do in the following Theorem and Corollary, give the answers to our remaining definitions.

Theorem 5.3.
The answer to (v) (1) is no.

Proof:
From Lemma 5.5 we know that Γ_ε satisfies (v). Lemma 5.6 shows that $\dim\Gamma_\varepsilon > 1$ and therefore that Γ_ε is a counter example to the answer to (v) (1) being yes. $\diamondsuit$

Corollary 5.3.
The answer to the following definitions is no.

(v) (2),

(ii) (1), and

(ii) (2).

Proof:
Since from Lemma 5.6 we know that the dimension of Γ_ε is greater than 1, it follows that Γ_ε cannot be weak locally $\mathcal{H}^1$-finite. Since Lemma 5.5 then shows that Γ_ε satisfies (v), it follows that the answer to (v) (2) must be no.

Since Property (v) is strictly stronger thatn Property (ii). Any set that satisfies (v) must also satisfy (ii). It then follows that Γ_ε satisfies (ii) and thus in the same way that the answer to (v) (1) and (2) is no it follows that the answers to (ii) (1) and (2) is no. $\diamondsuit$

This completes the classification results that were the inital motivating aim for this work. We present again here a summary of the classification results:

$$\begin{array}{rl}
(i) & (1)\ \ no\ \ (2)\ \ no \\
(ii) & (1)\ \ no\ \ (2)\ \ no \\
(iii) & (1)\ \ yes\ \ (2)\ \ no \\
(iv) & (1)\ \ yes\ \ (2)\ \ no \\
(v) & (1)\ \ no\ \ (2)\ \ no \\
(vi) & (1)\ \ yes\ \ (2)\ \ no \\
(vii) & (1)\ \ yes\ \ (2)\ \ yes(weak)/no(strong) \\
(viii) & (1)\ \ yes\ \ (2)\ \ yes.
\end{array}$$

We next continue with results related to the fitting of the counter examples to the eight properties. In particular we show that A_ε does indeed spiral in a sense that will be defined and we show that the counterexamples can be extended to higher dimensions.

We have already seen that a rich tapestry of results follows from these more complicated examples. In the interest of finding as much interesting mathematics as possible that could arise from these sets we then in Chapters 7 and 8 allow for generalisation of these sets and show various measure theoretic properties of the resulting sets.

Chapter 6

Miscellaneous Results

In this section we present some further interesting and relevant results found in association with the study leading to the classification that we have presented, but that were not directly necessary in the classification. In particular we show that our present counter examples would not be sufficient for a δ-fine version of property (v) and that A_ε does not satisfy (vii), showing that there is no direct strength ranking of the 8 definitions in Definition 2.2 since Λ_δ which satisfies (vii) does not satisfy (iii) which is satisfied by A_ε. Further,in the proof that A_ε does not satisfy (vii) we see that the sets A_ε do infact spiral infinitely finely in a sense that will become clearer.

We also discuss how to extend the presented counter examples into higher dimensional counter examples. We show one such extension since the process of extending to higher dimensions remains the same for each of the sets.

6.1 The Existence of Spiralling

We start by showing that A_ε does not satisfy (v) for each $\delta > 0$. Similarly, but oppositely to Lemma 5.5 we show that there is also a function $\delta_1 : \mathbb{R} \to \mathbb{R}$ such that for each $\varepsilon > 0$, for each $\delta < \delta_1(\varepsilon)$, A_ε does not satisfy (v) for δ. Thus, although for each δ there is an A_ε that fites, there is no ε such that A_ε satisfies (v) for every δ, thus showing that A_ε and indeed Γ_ε are not sufficient as counter examples to any δ-fine version of (v).

Proposition 6.1. *There is a function*

$$\delta_1 : \mathbb{R} \to \mathbb{R}^+$$

with $\delta_1(x) > 0$ for all $x > 0$ such that for each $\varepsilon > 0$ and all $\delta < \delta(\varepsilon)$ A_ε does not satisfy (v) with respect to δ.

Proof:
First, we take
$$y \in A_\varepsilon \cap B_{\frac{\varepsilon}{32}}\left(\left(\frac{1}{2}, 2\varepsilon\right)\right)$$
and $\rho > \frac{3}{8}$, say $\rho = \rho_0 = \frac{1}{2}$ ($\rho_0 = \frac{1}{2}$ as $B_{1/2}((1/2,0)) \supset A$. It is not hard to see that we then have
$$\partial B_\rho(y) \cap int(T_{3,1}) \neq \emptyset$$
and
$$\partial B_\rho(y) \cap int(T_{3,4}) \neq \emptyset$$
so that, more particularly
$$B_\rho(y) \cap T_{3,1} \cap A_\varepsilon \neq \emptyset$$
and
$$B_\rho(y) \cap T_{3,4} \cap A_\varepsilon \neq \emptyset.$$
We now note that
$$\sup_{x \in T_{3,i}} \pi_y(x) < \varepsilon$$
for $i = 1, 4$ and that clearly
$$\inf_{x \in B_{\frac{\varepsilon}{32}}((\frac{1}{2}, 2\varepsilon))} \pi_y(x) > \frac{63\varepsilon}{32}.$$

It follows that a vertical gap between y and points in $A \cap B_\rho(y)$ of atleast $\frac{31\varepsilon}{32}$ exists both "to the left" of y (that is points $z \in T_{3,1}$ where we must have $\pi_x(z) < \pi_x(y)$) and "to the right" of y (similar to above, that is points $z \in T_{3,4}$ where we must have $\pi_x(z) > \pi_x(y)$).

Similarly clearly, we know that $\pi_x(z) > 0$ for all $x \in T_{3,1}$ (and also in fact $T_{3,4}$) and conversely we have $\pi_x(z) < 1$ for all $z \in T_{3,4}$ (and in fact, but unimportantly $T_{1,4}$). Also we have
$$\begin{aligned} \pi_x(y) &\in \left(\frac{1}{2} - \frac{\varepsilon}{32}, \frac{1}{2} + \frac{\varepsilon}{32}\right) \\ &\subset \left(\frac{63}{128}, \frac{65}{128}\right) \end{aligned}$$
since $\varepsilon < \frac{1}{4}$.

This means that in the best case any cone has less than a horizontal length of $\frac{65}{128}$ to spread out to meet a set of vertical distance

$$\frac{31\varepsilon}{32}$$

away from it's center.

Supposing, to begin with, that $L_{y,\rho}||\mathbb{R}_x$ (that is $L_{y,\rho}$ is parallel to the horizontal axis) then the cone angle must be, to cover the "best case mentioned above" at least

$$tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right).$$

Now, should $L_{y,\rho}$ not be horizontal, we have that it is either positively or negatively sloped, but in either case, it continues to go through y. In the former case, we have that the cone angle estimate is imporoved for points in $T_{3,1}$ however, continuing to observe the $y = (63/128, 63\varepsilon/32)$ case with a $z \in T_{3,4}$, it is clear that the resultant required cone angle for this z can be no better than the cone angle required to include $z = (1, \varepsilon)$. We must therefore have that the minimum cone angle is no better than

$$\begin{aligned}\theta &= tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right) + ||(L_{y,\rho} - y) - \mathbb{R}_x||_{G(1,2)} \\ &> tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right)\end{aligned}$$

where $||\cdot||_{G(1,2)}$ denotes the norm on the grassman manifold of 1-planes in $\mathbb{R}^2$. A similar argument works considering points in $T_{3,1}$ in the case that $L_{y,\rho}$ is negatively sloped possibly improving the estimate for points in $T_{3,4}$. We therefore have that the cone angle

$$tan^{-1}\left(\frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}\right)$$

cannot be improved on, so that for any

$$\delta < \frac{\left(\frac{31\varepsilon}{32}\right)}{\left(\frac{65}{128}\right)}$$

A_ε cannot satisfy (v) with respect to δ.

Thus the function δ_1 defined by

$$\delta_1(x) := \frac{\left(\frac{31x}{32}\right)}{\left(\frac{65}{128}\right)}$$

satisfies the requirements for the Proposition. $\diamondsuit$

To prove that A_ε (and indeed Γ_ε) cannot satisfy (vii) irrespective of δ, we have to show that although no spiralling occurs in the vicinity of any given point in A_ε at a given approximation level, spiralling does indeed occur.

This means that for any point $x \in A_\varepsilon$, any radius $r > 0$ and any potential approximating affine space, there exists a (smaller) triangular cap in $B_r(x)$ whose base is arbitrarily close to perpendicular to the approximating affine space.

It then follows that an approriate choice of testing points and testing radius smaller than or equal to r in such a triangular cap will allow us to show that for any $\delta < 1$ A_ε and indeed Γ_ε cannot possibly satisfy (vii).

Proposition 6.2.
For each $\varepsilon > 0$ and $0 < \delta < 1$, A_ε does not satisfy Property (vii) with respect to δ.

Proof:
Let $\delta \in (0, 1)$, $\varepsilon > 0$ and $y \in A_\varepsilon$; then should A_ε satisfy the definition then for each $\rho_y > 0$ there would exist an affine space L_{y,ρ_y} such that for all $x \in A_\varepsilon \cap B_{\rho_y}(y)$ and all $\rho < \rho_y$ $B_\rho(x) \cap A_\varepsilon \subset L^{\delta\rho}_{y,\rho} + x$.

Now, since we are assuming that A_ε satisfies the definition there must be a function,

$$\phi : (0, 1) \to (0, 2\pi)$$

dependent only on δ which describes the cone outside of which no boundary points of a ball around a point $x \in B_{\rho_y}(y)$ are in A_ε. That is by defining

$$E_{\phi,\rho,x} := \left\{ x \in \partial B_\rho(x) : tan^{-1}\left(\frac{\pi^{\perp}_{L_{y,\rho_y}}(x)}{\pi_{L_{y,\rho_y}}(x)}\right) \geq \phi(\delta) \right\}$$

we have

$$A_\varepsilon \cap E_{\phi(\delta),\rho,x} = \emptyset,$$

for all $x \in A \cap B_{\rho_0}(y)$ and also that there is a $\eta(\delta) > 0$ such that for all $x \in A \cap B_{\rho_0}(y)$, $\rho \in (0, \rho_0]$ and all $z \in E_{\frac{\pi}{2} - \frac{\pi - \psi(\delta)}{2}, \rho, x}$

$$B_{\rho\eta(\delta)}(z) \cap A = \emptyset. \tag{6.1}$$

That is, around points in the central part of $E_{\phi,\rho,x}$ we can put a ball depending only on ρ and δ that will be completely empty of A_ε.

We observe that y must be in some trianglular cap of the construction of A, $T_{n,i}$, for some n and i, also such that $T_{n,i} \subset B_{\rho_0}(y)$. We make the nomenclatutorial choice to call the vertices of the triangular cap $T_{m,i}$ $a_{m,i}, l_{m,i}$, and $r_{m,i}$ chosen such that

$$\pi_x \circ O_{A_{m,i}}(a_{m,i}) = 0,$$

$$\pi_x \circ O_{A_{m,i}}(l_{m,i}) < 0, \text{ and}$$

$$\pi_x \circ O_{A_{m,i}}(r_{m,i}) > 0.$$

That is a denotes the "top" vertici as we have previously defined, and l and r denote the identical "left" and "right" base angles.

We now note that for each $k \in \mathbb{N}$ we have

$$\psi_{T_{n,i}+z(i,k)}^{T_{n+k,2^k i+4^{k-1}+2}} = \sum_{i=n}^{n+k} \theta_{i,\cdot}$$

for some appropriate point $z(i,k) \in \mathbb{R}^2$.

We now need some properties of the sequence $\{\theta_{i,\cdot}\}_{i=1}^{\infty}$. First of all we recall that

$$\lim_{i\to\infty} \theta_{i,\cdot} = 0. \tag{6.2}$$

And that we can specifically write that

$$\theta_{n,\cdot} = tan^{-1}\left(\frac{8\varepsilon}{(1+16n\varepsilon^2)^{1/2}}\right).$$

so that using the facts that

$$\frac{dtan}{dx}(x) \geq 0,$$

and

$$\frac{dtan}{dx}(x)|_{x=0} = 1$$

(and hence for sufficiently large n, $tan^{-1}(1/\varepsilon n) > 1/(2\varepsilon n)$), we get for any $n_0 \in \mathbb{N}$

$$\begin{aligned}
\sum_{n=n_0}^{\infty} \theta_{i,\cdot} &= \sum_{n=n_0}^{\infty} tan^{-1}\left(\frac{8\varepsilon}{(1+16n\varepsilon^2)^{1/2}}\right) \\
&\geq \sum_{n=n_0}^{\infty} tan^{-1}\left(\frac{8\varepsilon}{4(1+n\varepsilon^2)^{1/2}}\right) \\
&= \sum_{n=n_0}^{\infty} tan^{-1}\left(\frac{8\varepsilon}{4\varepsilon(\frac{1}{\varepsilon^2}+n)^{1/2}}\right) \\
&\geq \sum_{n=n_0}^{\infty} \frac{8\varepsilon}{8\varepsilon(\frac{1}{\varepsilon^2}+n)^{1/2}} \\
&= \sum_{n=n_0}^{\infty} \frac{1}{(\frac{1}{\varepsilon^2}+n)^{1/2}} \\
&\geq \sum_{n=n_0+E}^{\infty} \frac{1}{n^{1/2}} \\
&> \sum_{n=n_0+E}^{\infty} \frac{1}{n} \\
&= \infty,
\end{aligned}$$

where E denotes the smallest integer greater than or equal to $1/\varepsilon^2$. It follows that there exists a sequence, $\{n_k\}$, such that for each $k \in \mathbb{N}$

$$\sum_{i=n}^{n_k-1} \theta_{i,\cdot} < \frac{2k\pi - \pi}{2} < \sum_{i=n}^{n_k+1} \theta_{i,\cdot}.$$

So that there is a triangular cap $T_{n_k,i(k)}$ (for the appropriate i depending on k) such that

$$\begin{aligned}
tan^{-1}\left(\frac{\pi^{\perp}_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}{\pi_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}\right) &- \frac{2k\pi - \pi}{2} \\
&< \theta_{n_k-1,\cdot} + \theta_{n_k,\cdot}.
\end{aligned}$$

Thus, by (6.2) there exists a $k \in \mathbb{N}$ such that

$$\begin{aligned}
tan^{-1}\left(\frac{\pi^{\perp}_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}{\pi_{L_{y,\rho_0}}(r_{n_k,i(k)} - l_{n_k,i(k)})}\right) &- \frac{2k\pi - \pi}{2} \\
&< \theta_{n_k-1,\cdot} + \theta_{n_k,\cdot} \\
&< \frac{\frac{\pi}{2} - \phi(\delta)}{2}.
\end{aligned}$$

That is the triangular cap, $T_{n_k,i(k)}$ has the property that

$$r_{n_k,i(k)} \in E_{\phi(\delta)+\frac{\frac{\pi}{2}-\phi(\delta)}{2},|r_{n_k,i(k)}-l_{n_k,i(k)}|,l_{n_k,i(k)}}.$$

The endpoints themselves are not in A, however, we can choose $x_l, z_r \in A$ such that

$$|x_l - r_{n_k,i(k)}| < \frac{\eta(\delta)|r_{n_k,i(k)} - l_{n_k,i(k)}|}{2}$$

and

$$|z_r - l_{n_k,i(k)}| < \frac{\eta(\delta)|r_{n_k,i(k)} - l_{n_k,i(k)}|}{2}$$

so that there is a $x_r \in \partial B_{|r_{n_k,i(k)}-l_{n_k,i(k)}|}(x_l)$ such that

$$z_r \in B_{\eta(\delta)|r_{n_k,i(k)}-l_{n_k,i(k)}|}(x_r).$$

Since, by our choice of triangular cap, $T_{n,i}$, $x_l \in B_{\rho_0}(y)$ and $|r_{n_k,i(k)}-l_{n_k,i(k)}| < \rho_0$ this contradicts (6.1), proving the proposition since ε and δ were chosen arbitrarily. $\diamondsuit$

6.2 Higher Dimension Analogies of Γ_ε, A_ε and $\mathcal{A}_\varepsilon$

We now come to the higher dimensional generalisations of the counterexamples.

It is unfortunately trivial - unfortunate from the view of finding interesting mathematics - to generalise our counter examples to higher dimensions so that we obtain no further insite into how the structures of sets work. In each case we simply cross each set with either an interval or simply the plane of the required dimension, depending on whether or not we need the set to be bounded (as we do for ρ_0 uniformity properties). We show, as an example, how A_ε is extended, and demonstrate how it continues to satisfy Property (iii).

Suppose that we are taking j-dimensional approximations in $\mathbb{R}^{j+k}$. We take

$$S_\varepsilon = A_\varepsilon \times \mathbb{R}^{j-1} =\subset \mathbb{R}_A \times \mathbb{R}^{j-1} \times \mathbb{R}^{A_c},$$

where $\mathbb{R}_A = \mathbb{R}_{Ac} = \mathbb{R}$ but have been given names for notational convenience. $\mathbb{R}_A$ and $\mathbb{R}_{A_c}$ are identified with $\mathbb{R}$ and $\mathbb{R}^2/\mathbb{R}$ as we have been considering in

the preceeding sections so that $A_\varepsilon \subset \mathbb{R}_A \times \mathbb{R}_{A_c}$. Further S_ε is constructed inside of

$$\mathbb{R}^{j+k} = \mathbb{R}_A \times \mathbb{R}^{j-1} \times R_{A,c} \times \mathbb{R}^{k-1}.$$

We can thus see S_ε as

$$\begin{aligned} S_\varepsilon &= \{y = (y_1, x_2, ..., x_{n-1}, y_2, 0, ..., 0) : (y_1, y_2) \in A_\varepsilon \subset \mathbb{R}_A \times \mathbb{R}_{Ac}, x_i \in \mathbb{R}\} \\ &\subset \mathbb{R}_A \times \mathbb{R}_x^{j-1} \times R_{A_c} \times \mathbb{R}_z^{k-1}, \end{aligned}$$

where $\mathbb{R}_x^{j-1} \cong \mathbb{R}^{j-1}$ and $\mathbb{R}_z^{k-1} \cong \mathbb{R}^{k-1}$ are again notational conveniences denoting the dimensions along which the extension of A_ε into S_ε exist ($\mathbb{R}_x^{j-1}$), and the additional codimensions ($\mathbb{R}_z^{k-1}$).

Proposition 6.3.
S_ε shows that the answer to (iii) (2) is no for arbitrary j.

Proof:
There are two properties that we need to show that S_ε has. That it has the fine weak j-dimensional δ-approximation property, and that for each $x \in S_\varepsilon$ and $R > 0$, $B_R^{j+k}(x) = +\infty$.

First, to show that S_ε has property (iii). We take arbitrary $y \in S_\varepsilon$ and $\delta > 0$. We now need only show that there exists a j-dimensional affine space, $L_{y,\rho}$ for each $\rho > 0$, such that

$$S_\varepsilon \cap B_\rho(y) \subset L_{y,\rho}^{\delta\rho}.$$

We note that since A_ε has property (iii), there exists for the chosen δ and y a 1-dimensional affine space $L_{(y_1,y_2),\rho}$ such that

$$A_\varepsilon \cap B_\rho^2(y_1, y_2) \subset L_{(y_1,y_2),\rho}^{\delta\rho}.$$

It is therefore reasonable to take and test $L_{y,\rho} = L_{(y_1,y_2),\rho} \times \mathbb{R}_x^{j-1}$ as our affine space. Clearly

$$\begin{aligned} S_\varepsilon \cap B_\rho(y) &= (A_\varepsilon \cap \pi_{\mathbb{R}_A \times \mathbb{R}_{A_c}}(B_\rho(y))) \times \mathbb{R}_x^{j-1} \\ &\subset L_{(y_1,y_2),\rho}^{\delta\rho} \times \mathbb{R}_x^{j-1} \\ &= L_{y,\rho}^{\delta\rho}, \end{aligned}$$

which gives us that S_ε has the appropriate property.

To show that there is infinite measure in each ball $B_{R_1}(y)$ we take an $R_1 > 0$ and a $y \in S_\varepsilon$. Let $R = d(y, \partial S_\varepsilon)$. We then get that

$$\begin{aligned}
\mathcal{H}^j(S_\varepsilon \cap B_{R_1}(y)) &\geq \mathcal{H}^j(S_\varepsilon \cap B_R(y)) \\
&\geq \mathcal{H}^j(S_\varepsilon \cap ([-R/4, R/4]^{j+k} + (y_1, 0, ..., y_2, 0, ..., 0))) \\
&= \mathcal{H}^1(A \cap ([-R/4, R/4]^2 + (y_1, y_2)))\mathcal{H}^{j-1}(\pi_{R_x^{j-1}}) \\
&= \mathcal{H}^1(A \cap ([-R/4, R/4]^2 + (y_1, y_2))) \left(\frac{R}{4}\right)^{j-1} \\
&= +\infty,
\end{aligned}$$

showing that S_ε is not weak locally $\mathcal{H}^j$-finite. $\diamondsuit$

Chapter 7

Generalised Koch type sets and Relative centralisation of sets

We turn now to the generalisation of the sets A_ε and Γ_ε which in our generalisations turn out to be two examples of the same sort of set. As already hinted at in Definition 5.3 the generalisation can be seen as increasing the freedom with which the base angles of the triangular caps $\theta^A_{n,i}$ for a set A. We allow this freedom in two differing strengths. Firstly that $\theta_{n,i} = Tn, j$, $i, j \in \{1, ..., 2^n\}$ as in the construction of A_ε. Secondly that $\theta_{n,j}$ are allowed to vary freely over n and j. A common restriction to the two variations is that $T_{n,i} \subset T_{m,j} \Rightarrow \theta_{n,i} \leq \theta_{m,j}$. That is, as we take triangular caps inside of previously constructed ones, the base angles reduce. The rate of reduction in seperate triangular caps may of course vary.

It is clear that the second variation is a direct generalisation of the first. We keep them seperate however since the second allows more complications than the first and so some results are able to be presented in a stronger form for the first variation.

The original motivation for this investigation stems from an interest in the dimension of these sets. Γ_ε and A_ε are both examples of the first variation where for Γ_ε, $\theta^{\Gamma_\varepsilon}_{n,i}$ is constant over n and i, whereas $\theta^{A_\varepsilon}_{n,i}$ varies by strictly decreasing to 0 in n. The question being whether higher dimensions than (in this case) 1 could only be reached with constant base angle as in Γ_ε. The answer turns out to be no. Along with a presentation of this answer in both variations of our generalisation we present various other results concerning measure and rectifiability erlating to our generalisations.

In this chapter we present the two main definitions of the sets in question

and show their equivalence (both definitions will be used as which is more convenient in proofs that we present varies). We further show another characterisation of these sets in terms of a bijection from $\mathbb{R}$. We then present some general lemmas and background results necessary to present the main results concerning measure, rectifiability and dimension. The main results are then presented in the next and final chapter.

7.1 Equivalent Constructions of Koch Type Sets

We start, quite naturally with definitions, equivalences and characterisations. First of all with a formal definition of the first variation of the generalisations.

Definition 7.1.
Suppose we can construct a set B as follows:

Let $A_{0,1}$ be a base (a line in $\mathbb{R}^2$) and $T_{0,1}$ be a triangular cap on $A_{0,1}$ with vertical height $\varepsilon\mathcal{H}^1(A_{0,1})$ with $\varepsilon < 1/100$. Let $\theta_{0,\cdot}$ be the base angles of $T_{0,1}$ and the two shorter sides of $T_{0,1}$ be named $A_{1,1}$ and $A_{1,2}$. We then construct two new triangular caps $T_{1,1}$ and $T_{1,2}$ on $A_{1,1}$ and $A_{1,2}$ with base angles $\theta_{1,\cdot} \leq \theta_{0,\cdot}$. We define

$$A_0 = T_{0,1}$$

and

$$A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ a union of 2^n triangular caps with base angles $\theta_{n,\cdot}$ and 2^{n+1} shorter sides labelled $A_{n,i}$, $i \in \{1, ..., 2^{n+1}\}$. Then construct a triangular cap $T_{n+1,i}$ on each $A_{n+1,i}$ such that the base angles $\theta_{n+1,\cdot}$ satisfy $\theta_{n+1,\cdot} \leq \theta_{n,\cdot}$. Define $A_{n+1} = \cup_{i=1}^{2^{n+1}} T_{n,i}$. Finally define

$$B = \bigcap_{n=0}^{\infty} A_n.$$

*We then call a set A an A_ε-**type** set whenever $A \in \{B, B \sim E(B)\}$.*

Then immediately we define the second variation.

Definition 7.2.
Suppose we can construct a set B as follows:

Let $A_{0,1}$ be a base (a line in $\mathbb{R}^2$) (for our purposes, provided that the line

has non-infinite, non-zero length, it's position and length have no effect on the properties with respect to rectifiability, dimension, etc. and so without loss of generality we will generally assume that $A_{0,1} = [0,1] \subset \mathbb{R}$*) and* $T_{0,1}$ *be a triangular cap on* $A_{0,1}$ *with vertical height* $\varepsilon\mathcal{H}^1(A_{0,1})$ *with* $\varepsilon < 1/100$*. Let* θ_0 *be the base angle of* $T_{0,1}$ *and the two shorter sides of* $T_{0,1}$ *be denoted* $A_{1,1}$ *and* $A_{1,2}$*. We then construct two new triangular caps* $T_{1,1}$ *and* $T_{1,2}$ *on* $A_{1,1}$ *and* $A_{1,2}$ *with base angles* $\theta_{1,1}, \theta_{1,2} \leq \theta_0$*. We define*

$$A_0 = T_{0,1}$$

and

$$A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ *a union of* 2^n *triangular caps with base angles* $\theta_{n,i}$ *and* 2^{n+1} *"shorter sides" (two per triangular cap) labelled* $A_{n+1,i}$*,* $i \in \{1, ..., 2^{n+1}\}$*. Then construct a triangular cap* $T_{n+1,i}$ *on each* $A_{n+1,i}$ *such that the base angles* $\{\theta_{n+1,i}\}_{i=1}^{2^{n+1}}$ *satisfy for each* $i \in \{1, ..., 2^n\}$

$$\theta_{n,i} \geq \begin{cases} \theta_{n+1,2i-1} \\ \theta_{n+1,2i} \end{cases}.$$

(i.e. the new base angles for each triangular cap are bounded by the base angle of the nth level that the new triangular cap is contained in).

Define $A_{n+1} = \cup_{i=1}^{2^{n+1}} T_{n+1,i}$*. Finally define*

$$B = \bigcap_{n=0}^{\infty} A_n.$$

We then call a set A *a* **Koch type set** *whenever* $A \in \{B, B \sim E(B)\}$*. We denote the set of all such sets by* $\mathcal{K}$*.*

Remark: In general any notation that can be considered in relation to some set $A \in \mathcal{K}$, for e.g. $\theta_{n,j}$, $T_{n,j}$, etc., the superscript A will denote association with the set A when it may be unclear which set we are talking about. That is $T^A_{n,j}$ will denote the triangular cap $T_{n,j}$ associated with the construction of A.

Definition 7.3.
Let $A \in \mathcal{K}$*. Then*

$$\tilde{A}^A_n := \bigcup_{i=1}^{2^n} A^A_{n,i}$$

The second round of definitions for the two variations of generalisation are directly analogous to the original construction of A_ε in that we consider $\tilde{A}_{n,j}$ sets instead of the $T_{n,j}$ sets.

Definition 7.4.
Suppose we can construct a set B as follows:

Let $A_{0,1}$ be a base (a line in $\mathbb{R}^2$ of finite length) and $T_{0,1}$ be a triangular cap on $A_{0,1}$ with vertical height $\varepsilon\mathcal{H}^1(A_{0,1})$ with $\varepsilon < 1/100$. Let $\theta_{0,\cdot}$ be the base angles of $T_{0,1}$ and the two shorter sides of $T_{0,1}$ be named $A_{1,1}$ and $A_{1,2}$. We then construct two new triangular caps $T_{1,1}$ and $T_{1,2}$ on $A_{1,1}$ and $A_{1,2}$ with base angles $\theta_{1,\cdot} \leq \theta_{0,\cdot}$. We define

$$A_0 = T_{0,1}$$

and

$$A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ a union of 2^n triangular caps with base angles $\theta_{n,\cdot}$ and 2^{n+1} shorter sides labelled $A_{n,i}$, $i \in \{1, ..., 2^{n+1}\}$. Then construct a triangular cap $T_{n+1,i}$ on each $A_{n+1,i}$ such that the base angles $\theta_{n+1,\cdot}$ satisfy $\theta_{n+1,\cdot} \leq \theta_{n,\cdot}$. Define $\tilde{A}_{n+1} = \cup_{i=1}^{2^{n+1}} A_{n,i}$. Finally define

$$B = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} \tilde{A}_n.$$

*We then call a set A an A_ε-**type** set whenever $A \in \{B, B \sim E(B)\}$.*

Then immediately we define the second variation.

Definition 7.5.
Suppose we can construct a set B as follows:

Let $A_{0,1}$ be a base (a line in $\mathbb{R}^2$) (as previously, provided that the line has non-infinite, non-zero length, it's position and length have no effect on the properties with respect to rectifiability, dimension, etc. and so without loss of generality we will generally assume that $A_{0,1} = [0,1] \subset \mathbb{R}$) and $T_{0,1}$ be a triangular cap on $A_{0,1}$ with vertical height $\varepsilon\mathcal{H}^1(A_{0,1})$ with $\varepsilon < 1/100$. Let θ_0 be the base angle of $T_{0,1}$ and the two shorter sides of $T_{0,1}$ be denoted $A_{1,1}$ and $A_{1,2}$. We then construct two new triangular caps $T_{1,1}$ and $T_{1,2}$ on $A_{1,1}$ and $A_{1,2}$ with base angles $\theta_{1,1}, \theta_{1,2} \leq \theta_0$. We define

$$A_0 = T_{0,1}$$

and

$$A_1 = \bigcup_{i=1}^{2} T_{1,i}.$$

Then suppose we have $A_n = \cup_{i=1}^{2^n} T_{n,i}$ *a union of* 2^n *triangular caps with base angles* $\theta_{n,i}$ *and* 2^{n+1} *"shorter sides" (two per triangular cap) labelled* $A_{n+1,i}$, $i \in \{1, ..., 2^{n+1}\}$. *Then construct a triangular cap* $T_{n+1,i}$ *on each* $A_{n+1,i}$ *such that the base angles* $\{\theta_{n+1,i}\}_{i=1}^{2^{n+1}}$ *satisfy for each* $i \in \{1, ..., 2^n\}$

$$\theta_{n,i} \geq \begin{cases} \theta_{n+1,2i-1} \\ \theta_{n+1,2i} \end{cases}.$$

(i.e. the new base angles for each triangular cap are bounded by the base angle of the nth level that the new triangular cap is contained in).

Define $\tilde{A}_{n+1} = \cup_{i=1}^{2^{n+1}} A_{n+1,i}$. *Finally define*

$$B = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} \tilde{A}_n.$$

We then call a set A *a* **Koch type set** *whenever* $A \in \{B, B \sim E(B)\}$. *We denote the set of all such sets by* $\mathcal{K}$.

Definition 7.6. *Let* $A \in \mathcal{K}$ *we then define the edge points of* A, $E(A)$ *by*

$$E(A) := \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} E(T_{n,i}^A)$$

where $E(T_{n,i}^A)$ *is as defined in Definition 3.2.*

Before going on to show that these definitions are equivalent we need the following simple but important fact.

Lemma 7.1.
Let $A \in \mathcal{K}$. *Then for any sequence* $\{n, i(n)\}_{n\in\mathbb{N}}$ *such that* $T_{n,i(n)} \subset T_{n-1,i(n-1)}$ *for each* $n \in \mathbb{N}$

$$\lim_{n\to\infty} \mathcal{H}^1(A_{n,i(n)}) = 0.$$

Proof:
Since, by assumption $\theta_{0,1} < \pi/32$ and by construction $\theta_{n,i(n)}$ is decreasing in n. It follows from the inductive definition of the $A_{n,i(n)}$'s that

$$\begin{aligned} \mathcal{H}^1(A_{n,i(n)}) &= (cos\theta_{n-1,i(n-1)})^{-1}\mathcal{H}^1(A_{n-1,i(n-1)}) \\ &\leq (cos\theta_{0,1})^{-1}\mathcal{H}^1(A_{n-1,i(n-1)}) \\ &= C\mathcal{H}^1(A_{n-1,i(n-1)}) \end{aligned}$$

where $C = (cos\theta_{0,1})^{-1} < 1$. It follows inductively that

$$\mathcal{H}^1(A_{n,i(n)}) \leq C^n \mathcal{H}^1(A_{0,1}).$$

Since $\mathcal{H}^1(A_{0,1}) < \infty$ by construction, the result follows. $\diamondsuit$

We now show that these definitions are equivalent.

Proposition 7.1.
Definition 7.1 is equivalent to Definition 7.4. Definition 7.2 is equivalent to Definition 7.5.

Proof:
We show these equivalences by showing that should $\mathcal{A}_2$ be defined as in Definition 7.2 and $\mathcal{A}_1$ be defined as in Definition 7.5 with the same $T_{n,i}$, $A_{n,i}$, $\theta_{n,i}$ etc. then

$$\mathcal{A}_1 = \overline{\bigcup_{n=0}^{\infty} \tilde{A}_n} \sim \bigcup_{n=0}^{\infty} \tilde{A}_n = \left(\bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} T_{n,i} \right) - E(A) = \mathcal{A}_2 - E(\mathcal{A}_2) = \mathcal{A}_2 - E(\mathcal{A}_1).$$

That $E(\mathcal{A}_1) = E(\mathcal{A}_2)$ follows from Definition 7.6 and the fact that the $T_{n,i}$ used for $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same. We thus denote $E(A) := E(\mathcal{A}_1) = E(\mathcal{A}_2)$. This will complete the proof since $E(A)$ is countable and thus $\mathcal{H}^1(E(A)) = 0$.

As in Lemma 3.1 we see that

$$\mathcal{A}_1 + E(A)$$

is closed. Let

$$x \in \mathcal{A}_2 - E - \mathcal{A}_1.$$

then $d_x := d(x, \mathcal{A}_1 + E) > 0$.

Now, for each $n \in \mathbb{N}$, $x \in T_{n,i}$ for some i so that

$$d(x, \mathcal{A}_1 + E) < diam(T_{n,i}) = \mathcal{H}^1(A_{n,i}).$$

From Lemma 7.1 we have

$$\lim_{n \to \infty} \mathcal{H}^1(A_{n,i}) = 0.$$

Hence there is an $n_0 \in \mathbb{N}$ such that $diam(T_{n_0,j}) = \mathcal{H}^1(A_{n_0,j}) < d_x$ which implies

$$d(x, \mathcal{A}_1 + E) < \mathcal{H}^1(A_{n_0,i}) < d_x = d(x, \mathcal{A}_1 + E).$$

This contradiction implies

$$\bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n} T_{n,i} \subset \mathcal{A}_1 + E$$

and thus that

$$\left(\bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n} T_{n,i}\right) - E(A) \subset \mathcal{A}_1.$$

Next, it is clear from definition that

$$\bigcup_{i=1}^{2^n} A_{n,i} \subset \bigcup_{i=1}^{2^n} T_{n,i}$$

so that

$$\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{2^n} A_{n,i} \subset \bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n} T_{n,i} = \mathcal{A}_2$$

and thus, since $\mathcal{A}_2 = \cap_{n=0}^{\infty} \cup_{i=1}^{2^n} T_{n,i}$ is closed

$$\overline{\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{2^n} A_{n,i}} \subset \mathcal{A}_2.$$

Hence

$$\mathcal{A}_1 \subset \overline{\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{2^n} A_{n,i}} - E(A) \subset \mathcal{A}_2 - E(A).$$

Therefore,

$$\mathcal{A}_1 = \mathcal{A}_2 - E(A).$$

$\diamondsuit$

Before moving on to the further characterisations of these sets we present another useful equivalence of representation concerning the constructional pieces of sets in $\mathcal{K}$

Proposition 7.2.
For any $A \in \mathcal{K}$ and any $\xi \in \mathbb{R}^+$

$$\bigcup_{n=0}^{\infty} \bigcap_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)} = \bigcup_{x\in\Lambda_{\xi+}} \bigcap_{n=0}^{\infty} T_{n,i(n,x)}.$$

Proof:
First, suppose

$$z \in \bigcup_{n=0}^{\infty} \bigcap_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)}.$$

Then, since $\theta_{n,i(n,x)}$ is decreasing in n for all $x \in A$, so that $\theta_{n,i(n,x)} \geq \xi$ for all $n \in \mathbb{N}_0$ and $x \in \Lambda_{\xi+}$, and since for all $n \in \mathbb{N}_0$ $zinT_{n,i(n,x0}$ for some $x \in \Lambda_{\xi+}$ it follows that $\theta_{n,i(n,z)} \geq \xi$ for each $n \in \mathbb{N}_0$ and thus

$$\lim_{n\to\infty} \theta_{n,i(n,z)} \geq \xi$$

so that $z \in \Lambda_{\xi+}$.

Since clearly $z \in T_{n,i)n,z)}$ for each $n \in \mathbb{N}_0$ we can write

$$\bigcup_{x\in\Lambda_{\xi+}} \bigcap_{n=0}^{\infty} T_{n,i(n,x)} \supset \bigcup_{n=0}^{\infty} \bigcap_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)} \ni z,$$

so that

$$\bigcup_{n=0}^{\infty} \bigcap_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)} \subset \bigcup_{x\in\Lambda_{\xi+}} \bigcap_{n=0}^{\infty} T_{n,i(n,x)}.$$

For the other direction, suppose

$$z \in \bigcup_{x\in\Lambda_{\xi+}} \bigcap_{n=0}^{\infty} T_{n,i(n,x)}.$$

Then for some $x \in \Lambda_{\xi+}$ $z \in \cap_{n=0}^{\infty} T_{n,i(n,x)}$ and therefore

$$\begin{aligned} z &\in \bigcap_{n=0}^{\infty} T_{n,i(n,x)} \\ &\subset \bigcap_{n=0}^{\infty} \left(T_{n,i(n,x)} \cup \bigcup_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)} \right) \\ &= \bigcap_{n=0}^{\infty} \bigcup_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)}, \end{aligned}$$

so that

$$\bigcup_{x\in\Lambda_{\xi+}} \bigcap_{n=0}^{\infty} T_{n,i(n,x)} \subset \bigcup_{n=0}^{\infty} \bigcap_{i(n,x):x\in\Lambda_{\xi+}} T_{n,i(n,x)}.$$

Combining these two inclusions gives the result. $\diamondsuit$

7.2 Bijective Characterisation of Koch Type Sets

We now show that sets in $\mathcal{K}$ can be characterised by a bijection from $\mathbb{R}$ into $\mathbb{R}^2$. Since some sets in $\mathcal{K}$ do not have dimension 1 it may seem odd at first glance that such a bijection exists. By quoting the fact that there is a bijection between $\mathbb{R}$ and the Cantor set, however, we see that the concept is neither new or foreign in mathematics.

We show also immediately that a certain level of control of the preimage can retained. To this end we need the following definition.

Definition 7.7.
Let $A \in \mathcal{K}$ and $n \in \mathbb{N}$, then the dyadic interval of order n in $A_{0,0}$ (or simply, dyadic intervals of order n) are defined as the intervals D_n of the form

$$D_n = [l_{0,0} + i2^n(r_{0,0} - l_{0,0}), l_{0,0} + (i+1)2^n(r_{0,0} - l_{0,0})]$$

for some $i \in \{0, ..., 2^n - 1\}$. For some chosen $j \in \{0, ..., 2^n - 1\}$, the particular interval $D^A_{n,j}$ is defined as

$$D^A_{n,j} = [l_{0,0} + j2^n(r_{0,0} - l_{0,0}), l_{0,0} + (j+1)2^n(r_{0,0} - l_{0,0})].$$

As per usual the superscript A is dropped when the set is understood.

Remark: Note that should $A_{0,0}$ be, or be normed to be $[0, 1]$ on the real line, then the dyadic intervals in $A_{0,0}$ are simply the usual dyadic intervals.

Proposition 7.3.
Let $A \in \mathcal{K}$. Then there exists a sequence of Lipschitz functions $F_n : \mathbb{R} \mapsto \mathbb{R}^2$ (F^A_n when which set F_n is related to is not clear from the context) such that

$$F_n(A_{0,1}) = \tilde{A}_{n-1}.$$

Further there exists a bijection $\mathcal{F}$ ($\mathcal{F}^A$ when which set $\mathcal{F}$ is related to is not clear from the context) such that

$$\mathcal{F}(A_{0,1}) = A.$$

Additionally, denoting the relatively dyadic points of $A_{0,1}$ by D;
that is, for $\{x_1, x_2\} = E(A_{0,1})$, $x_1 < x_2$,

$$D := \{y : y = x_1 + (x_2 - x_1)j2^{-n}, n \in \mathbb{N}, j \in \{0, ..., 2^n\}\};$$

we have

$$\mathcal{F}(D) = E(A).$$

Finally for each dyadic interval $D_{n,i}$ in $A_{0,0}$,

$$F_n(D_{n,i}) = A_{n,i}$$

and

$$\mathcal{F}(D_{n,i}) \subset T_{n,i}.$$

Proof:
Since the proof is the same for any $A_{0,1}$, we assume for notational convenience that $A_{0,1} = [0,1]$. In this case D is also exactly the set of dyadic rationals in $[0,1]$. That is

$$D = \{j2^{-n} : n \in \mathbb{N}, j \in \{0,...,2^n\}\}.$$

We will define $\mathcal{F}$ as the limit of the F_n functions, and then show that it is well defined and has the required properties. Firstly, we define $f_0 : A_0 \to \mathbb{R}^2$ as

$$f_0(y) = \begin{cases} (y, tan\theta_{0,1}y) & y \in [0,1/2) \\ (y, tan\theta_{0,1}(1-y)) & y \in [1/2,1] \end{cases}.$$

We see clearly that f_0 is a Lipschitz bijection between A_0 and A_1 (Since the graph of the function draws out the triangular cap $T^A_{0,1}$) with Lipschitz constant (and Jacobian) $Lipf_0 = Jf_0 \equiv cos\theta_{0,1}^{-1}$. We then similarly define for each $n \in \mathbb{N}$ $f_{n,i} : A_{n,i} \to \mathbb{R}^2$ by

$$f_{n,i}(y) = \begin{cases} O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_{n,i}(\pi_x(O_{A_{n,i}}(y)) + \mathcal{H}^1(A_{n,i})/2)) & y \in I_1 \\ O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_{n,i}(\mathcal{H}^1(A_{n,i})/2 - \pi_x(O_{A_{n,i}}(y)))) & y \in I_2 \end{cases},$$

where $I_1 = O^{-1}_{A_{n,i}}([-\mathcal{H}^1(A_{n,i})/2, 0))$ and $I_2 = O^{-1}_{A_{n,i}}([0, \mathcal{H}^1(A_{n,i})/2])$. (Note that the $(1-y)$ factor in the definition of f_0 would change to some other appropriate constant should $A_{0,1} \neq [0,1]$.) We note in particular that $f_{n,i}(A_{n,i}) \subset T_{n,i}$. Noting also that the two end points of $A_{n,i}$ stay fixed we can define $f_n : A_n \to \mathbb{R}^2$ by

$$f_n(y) = f_{n,i}(y) \quad y \in A_{n,i}.$$

We see then that similarly to the f_0 situation f_n is a Lipschitz bijection between A_n and A_{n+1} with Lipschitz constant (and Jacobian in the case A is an A_ε type set) $Lipf_n (= Jf_n) = \max_{1 \le i \le 2^n} cos\theta_{n,i}^{-1}$.

By writing for a collection of functions $\{g_i\}_{i=0}^n$

$$\circ_{i=0}^n g_i = g_n \circ g_{n-1} \circ ... \circ g_0$$

we can then define the Lipschitz bijection between A_0 and A_{n+1}, $F_n : A_0 \to \mathbb{R}^2$ by

$$F_n = \circ_{i=0}^n f_i$$

which will then have Lipschitz constant (and Jacobian in the A_ε type set case)

$$LipF_n = JF_n = \prod_{i=1}^{n} (\cos\theta_i)^{-1}.$$

This demonstrates the first claim.

We can then propose a definition for $\mathcal{F}$ and indeed we propose the definition of $\mathcal{F} : A_0 \to \mathbb{R}^2$ to be

$$\mathcal{F}(y) = \lim_{n\to\infty} F_n(y).$$

We need first of all to show that this function is well defined. To do this we suppose first of all that

$$F_n(y) \in A_{n+1,i} \subset T_{n+1,i},$$

for some $i \in \{1, ..., 2^{n-1}\}$. Then

$$F_{n+1}(y) = f_{n+1,i}(y) \subset T_{n+1,i}.$$

Thus by induction, for each $n, k \in \mathbb{N}$

$$F_n(y) \in T_{n+1,i} \Rightarrow F_{n+k}(y) \in T_{n+1,i}$$

Then, From Lemma 7.1, since $diam(T_{n,i}) = \mathcal{H}^1(A_{n,i})$, $diam(T_{n,i(n)}) \to 0$ as $n \to \infty$ for any sequence $\{n, i(n)\}_{n\in\mathbb{N}}$ and thus by setting the sequence $\{i(y, n)\}_{n\in\mathbb{N}}$ to be the sequence such that $y \in T_{n,i(y,n)}$ for each $n \in \mathbb{N}$ (so that it is always well defined, we choose arbitrarily $i(n)$ to be chosen such that $y = l_{n,i}$ for each n for which y is an edge point) it follows that for any $\varepsilon > 0$ there is an $n_0 > 0$ such that for all $n, m = n + k > n_0$,

$$d(F_n(y), F_m(y)) < diamT_{n_0+1,i(y,n)} < \varepsilon$$

so that $\{F_n(y)\}$ is a Cauchy sequence in $\mathbb{R}^2$ and thus converges. It follows that $\mathcal{F}$ is well defined.

We need still to show that $\mathcal{F}$ is a bijective function between A_0 and A.

We note firstly that for any $y \in A_0$ $F_n(y) \in A_n$ so that $\mathcal{F}(y) \in \overline{\cup_{n=0}^{\infty} A_n}$ and thus

$$\mathcal{F}(A_0) = \bigcup_{y\in A_0} \mathcal{F}(y) \subset \overline{\bigcup_{n=0}^{\infty} A_n}.$$

Now, since new edge points $a_{n,i}$ are by the definition of triangular caps always directly over the center of the base of the triangular cap, it follows that for all $e \in E$, $e = a_{n,i}$ for some $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$ and thus $e = F_n((2i-1)/2^{n+1})$. Since the set $\{(2i-1)/2^{n+1})\}_{n\in\mathbb{N}, i\in\{1,...,2^n\}} = D$ the set of dyadic rationals, it follows that $\mathcal{F}(D) = E$ which is a claim in our Proposition.

Further, for all $y \in A_0 \sim D$, $F_{n+k}(y) \cap A_n \subset (A_{n+1+k} \sim E) \cap A_n = \emptyset$ for each $k \geq 0$ so that $\mathcal{F}(y) \notin A_n$ for all $n \in \mathbb{N}$. It thus follows that

$$\mathcal{F}(A_0 \sim D) \subset \overline{\bigcup_{n=0}^{\infty} A_n} - \bigcup_{n=0}^{\infty} A_n \in \{A, A - E(A)\} \subset A$$

and thus that

$$\mathcal{F}(A_0) = \mathcal{F}(A_0 \sim D) \cup \mathcal{F}(D) \subset A \cup E = A.$$

We therefore have $\mathcal{F} : A_0 \to A$. We now need to show that it is bijective. We first show, however, the final two claims that refer to the relationship of $\mathcal{F}$ to the dyadic intervals of $A_{0,0}$.

We quickly mention a sketch of a proof and motivation of the last two claims which will be more rigorously proven in the following result.

From the above comment on the image of the dyadic rationals and the definition of F_n for an $n \in \mathbb{N}$ it follows that for each $i \in \{1, ..., 2^{n+1}\}$

$$F_n\left(\left[\frac{i-1}{2^{n-1}}, \frac{i}{2^n}\right]\right) = A_{n+1,i}.$$

This proves also our second last claim. Since, we have from definition that from each $n \in \mathbb{N}$ and any $x \in A_{0,0}$, $F_{n+1}(x)$ is in the same triangular cap $T_{n,i}$ as $F_n(x)$. It follows from induction that $\mathcal{F}(x) \in T_{n,i}$. Since this is true for each x_0 such that $F_n(x_0) \in T_{n,i}$ and from the above this set is equal to $D_{n,i}$. It follows that $\mathcal{F}(D_{n,i}) \subset T_{n,i}$ which is our final claim in the Theorem.

Continuing with the proof of bijective we use the above proven important facts as follows.

Firstly, that should $x, z \in A_0$ with $x \neq z$ we then have that there is an $n \in \mathbb{N}$ such that $2^{1-n} \geq |x - z| > 2^{-n}$ and thus there exist $i, j \in \{1, ..., 2^{n+2}\}$ with $4 \geq |i - j| \geq 2$ and the property that $x \in [(i-1)2^{-n-2}, i2^{n-2}]$ and

$z \in [(j-1)2^{-n-2}, j2^{n-2}]$.

It then follows that $F_n(x) \in T_{n+2,i}$ and thus, as above, $\mathcal{F}(x) \in T_{n+2,i}$. Similarly $\mathcal{F}(j) \in T_{n+2,j}$.

Since from Lemma 5.1 we know that for any $n \in \mathbb{N}$ $T_{n+2,i} \cap T_{n+2,j} = \emptyset$ whenever $4 \geq |i-j| \geq 2$ it follows that $\mathcal{F}(x) \neq \mathcal{F}(z)$ and therefore that $\mathcal{F}$ is injective.

For surjectivity, we consider an arbitrary element $y \in A$. For all $n \in \mathbb{N}$, $y \in T_{n,i(y,n)}$ for some $i(y,n) \in \{1, ..., 2^n\}$. Then, again from

$$F_n\left(\left[\frac{i-1}{2^{n-1}}, \frac{i}{2^n}\right]\right) = A_{n+1,i}$$

we see that it is instructive to consider the intervals

$$\mathcal{F}^{-1}(A_{n,i(y,n)}) = [(i(y,n)-1)2^{-n}, i(y,n)2^{-n}] =: D_{n,i(y,n)}.$$

Since $T_{n+1,i(y,n)} \subset T_{n,i(y,n)}$ for each n it follows that $D_{n+1,i(y,n+1)} \subset D_{n,i(y,n)}$ for each n. We now observe $y_0 = \cap_{n=0}^{\infty} D_{n,i(y,n)}$. For this y_0

$$F_n(y_0) \subset F_n(D_{n,i(y,n)}) \subset A_{n,i(y,n)} \subset T_{n,i(y,n)}$$

for each n. Thus for each $n \in \mathbb{N}$, $|F_n(y_0) - y| \leq diam T_{n,i(y,n)}$. Since this diameter goes to zero as n approached infinity it follows that

$$\mathcal{F}(y_0) = \lim_{n\to\infty} F_n(y_0) = y.$$

From well definedness and the arbitrariness of y the surjectivity and thus bijectivity of $\mathcal{F}$ follows. $\diamondsuit$

We now show some results on the structure of $\mathcal{F}$ which expand on the last two points of the previous results, as well as embellsihing the proof somewhat. We show that the function can be looked at as a function on each dyadic interval. A in any given triangular cap is a bijection between A in this cap and a dyadic interval in $A_{0,0}$. These results make it much easier to track images and pre-images and thus also to track how much measure has come from, or gone to where.

Proposition 7.4.
Let $A \in \mathcal{K}$ be constructed from a base $[0,1]$. Then when $\{F_n\}_{n=0}^{\infty}$ are the Lipschitz functions such that

$$\mathcal{F}^A := \lim_{n\to\infty} F_n$$

pointwise on $A^A_{0,0}$ *and*

$$F_n = \circ^n_{i=0} f_i$$

and writing l^A_{ni}, a^A_{ni} *and* r_{ni} *as the edge point of* $A^A_{n,i}$ *adjoining* $A^A_{n,i-1}$ *(or (0,0) should* $i = 1$*), the centerpoint of* $A^A_{n,i}$ *and the edge point of* $A^A_{n,i}$ *adjoining* $A^A_{n,i+1}$ *(or (1,0) should* $i = 2^n$*) respectively.*

Then for $n \in \mathbb{N}$ *and* $i \in \{1, ..., 2^n\}$ *we have*

$$A^A_{n,i} = F_{n-1}([(i-1)2^{-n}, i2^{-n}]), F^{-1}_{n-1}(A^A_{n,i}) = [(i-1)2^{-n}, i2^{-n}],$$

$$l_{ni} = F_{n-1}(2^{-n}(i-1))$$

$$r_{ni} = F_{n-1}(2^{-n}i)$$

and that F_{n-1} *preserves relative distances. That is for each* $x, y \in [(i-1)2^{-n}, i2^{-n}]$

$$|F_{n-1}(x) - F_{n-1}(y)| = p_{n,i}|x-y|$$

for some $p_{n,i} \in \mathbb{R}$.

Remark: *Of the claims stated we are most interested in and thus emphasise*

$$A^A_{n,i} = F_{n-1}([(i-1)2^{-n}, i2^{-n}]), F^{-1}_{n-1}(A^A_{n,i}) = [(i-1)2^{-n}, i2^{-n}],$$

which gives us in essence a trace of the movement of a dyadic interval as it approaches the limit set A. With this we can follow the track either forward or backwards to identify which parts of A or $A_{0,0}$ *have positive measure given information about the measure of the other of A and* $A_{0,0}$*. The other claims are stated here as an aid to proving the inductive step which is the key to the proof.*

Proof:
We prove the statement by induction on n.

From the definition of $A^A_{1,1}$, $A^A_{1,2}$ and the definition

$$f_0(y) = \begin{cases} (y, tan\theta_0 y) & y \in [0, 1/2) \\ (y, tan\theta_0(1-y)) & y \in [1/2, 1] \end{cases}.$$

it follows that $A^A_{1,1} = F_0([0, 1/2]), A^A_{1,2} = F_0([1/2, 1])$, that $F_0^{-1}(A^A_{1,1}) = [0, 1/2], F_0^{-1}(A^A_{1,2}) = [1/2, 1]$, that $F_0(0) = (0, 0) = l_{11}$, that $F_0(1) = (1, 0) = r_{12}$, and hence that $F_0(1/2) = r_{11} = l_{12}$.

We see also that the preservation of relative distances holds with $p_{1,i} \equiv tan\theta^A_{0,1}$

for $i = 1, 2$. The claim thus holds for $n = 1$.

Now suppose that the claim is true for each $n \leq m$ for some $m \in \mathbb{N}$.

We note that for any arbitrary $i \in \{1, ..., 2^{m+1}\}$ there is a $j \in \{1, ..., 2^m\}$ such that $i \in \{2j-1, 2j\}$. Now since $A^A_{m,j} = F_{m-1}([(j-1)2^{-m}, j3^{-1}])$

$$\begin{aligned} F_m([(j-1)2^{-m}, j2^{-m}]) &= f_m \circ F_{m_1}([(j-1)2^{-m}, j2^{-m}]) \\ &= f_m(A_{m,j}). \end{aligned}$$

Since $F_{m-1}((j-1)2^{-m}) = l^A_{lmj}$, $F_m(j2^{-m}) = l^A_{rmj}$ and F_m preserves relative distances we also have

$$F_{m-1}((j-1)2^{-m} + 2^{-m-1}) = l^A_{mj},$$

and thus $\pi_x(O_{A^A_{n,j}}(F_{m-1}((j-1)2^{-m}+2^{-m-1}))) = 0$. Thus, again from relative distance preservation

$$\pi_x(O_{A^A_{n,j}}(F_{m-1}([(j-1)2^{-m}, (j-1)2^{-m} + 2^{-m-1}])) = [-\mathcal{H}^1(A^A_{n,j})/2, 0] =: I_1$$

and

$$\pi_x(O_{A^A_{n,j}}(F_{m-1}([(j-1)2^{-m} + 2^{-m-1}, j2^{-m}])) = [0, \mathcal{H}^1(A^A_{n,j})/2] =: I_2.$$

It follows then from the definition of $f_m|_{[(j-1)2^{-m}, j2^{-m}]}$

$$f_{n,i}(y) = \begin{cases} O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_n(\pi_x(O_{A_{n,i}}(y)) + \mathcal{H}^1(A_{n,i})/2)) & y \in I_1 \\ O^{-1}_{A_{n,i}}(\pi_x(O_{A_{n,i}}(y)), tan\theta_n(\mathcal{H}^1(A_{n,i})/2 - \pi_x(O_{A_{n,i}}(y)))) & y \in I_2 \end{cases},$$

and the definition of $A^A_{m+1,k}$, $k \in \{1, ..., 2^{-m-1}\}$ that

$$A^A_{m+1,2j-1} = F_m([(2j-2)2^{-m-1}, (2j-1)2^{-m-1}])$$

$$A^A_{m+1,2j} = F_m([(2j-1)2^{-m-1}, 2j2^{-m-1}])$$

and since we know F_m is a bijection

$$F^{-1}_m(A^A_{m+1,2j-1}) = [(2j-2)2^{-m-1}, (2j-1)2^{-m-1}]$$

$$F^{-1}_m(A^A_{m+1,2j}) = [(2j-1)2^{-m-1}, 2j2^{-m-1}]$$

Further:

$$F_m((2j-1)2^{-m-1}), F_m((2j-2)2^{-m-1}) \in E(A^A_{m+1,2j-1})$$

and

$$F_m((2j-1)2^{-m-1}), F_m(2j2^{-m-1}) \in E(A^A_{m+1,2j})$$

from which it must therefore follow that

$$F_m((2j-1)2^{-m-1}) = r^A_{(m+1)(2j-1)} = l^A_{l(m+1)(2j)}$$

$$F_m((2j-2)2^{-m-1}) = l^A_{(m+1)(2j-1)}$$

$$F_m(2j2^{-m-1}) = r^A_{(m+1)(2j)}.$$

Further since $F_{m-1}|_{[(2j-1)2^{-m},2j2^{-m}]}$ preserves relative distance with $|F_{m-1}(x) - F_{m-1}(y)| = p_{m-1,j}|x-y|$ for all $x, y \in [(2j-1)2^{-m}, 2j2^{-m}]$ from the definition of $f_m(y)$ and $F_m = f_m \circ F_{m-1}$ it follows that F_m preserves relative distances on $[(2j-2)2^{-m-1}, (2j-1)2^{-m-1}]$ and $[(2j-1)2^{-m-1}, 2j2^{-m-1}]$ with

$$p_{m,2j-1} = p_{m,2j} = (tan\theta^A_{n,j})p_{m-1,j}.$$

By substituting in i for $2j-1$ or $2j$ as necessary it follows that all required preoperties are satisfied for $m+1$ with the choice of $i \in \{1, ..., 2^{m+1}\}$. Since the choice of i was arbitrary this completes the inductive step and thus the proof. $\diamondsuit$

7.3 Further Characterisations and Properties of Sets in $\mathcal{K}$

Equiped with these results we are able to give a list of nomenclaturial definitions that will be instrumental in describing our results.

Definition 7.8.
Let $A \in \mathcal{K}$, then we write

$$\tilde{\theta}^A_{x_0} = \tilde{\theta}^A_x = \lim_{n\to\infty} \theta^A_{n,i(n,x)}$$

and define the functions $\tilde{\Pi}^A, \tilde{\Pi}^A_n, \tilde{\Pi}^A_{n,i} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\Pi}^A(x) = \prod_{i=0}^{\infty}(cos\theta^A_{n,i(n,x)})^{-1}$$

$$\tilde{\Pi}^A_n(x) = \prod_{i=0}^{n}(cos\theta^A_{n,i(n,x)})^{-1}$$

and

$$\tilde{\Pi}^A_{n,i} = \tilde{\Pi}^A_n(x),$$

for any $x \in A \cap T^A_{n,i}$. The superscript A is dropped when the set A is understood.

Further

$$\Lambda_m := \{x \in A : \tilde{\Pi}(x) \leq m\}$$
$$\Lambda^{-1}_m := \mathcal{F}^{-1}(\Lambda_m)$$
$$\Lambda_{m+} := \{x \in A : \tilde{\Pi}(x) \geq m\}$$
$$\Lambda^{-1}_{m+} := \mathcal{F}^{-1}(\Lambda_{m+})$$
$$\Lambda_\infty := \{x \in A : \tilde{\Pi}(x) = \infty\}$$
$$\Lambda^{-1}_\infty := \mathcal{F}^{-1}(\Lambda_\infty)$$

Also we introduce

$$i(n,x) := \mathbb{N} \times A_0 \rightarrow \mathbb{N}$$

defined by

$$i(n,x) := \{i \in \{1, ..., 2^n\} : x \in T^A_{n,i}\}$$

Also, we define for each $a \in \mathbb{R}$

$$\Upsilon^{-1}_a := \{x \in A_0 : \tilde{\theta}^A_x \leq a\}$$
$$\Upsilon_a := \mathcal{F}(\Upsilon^{-1}_a)$$
$$\Upsilon^{-1}_{a+} := \{x \in A_0 : \tilde{\theta}^A_x \geq a\}$$

and

$$\Upsilon_{a+} := \mathcal{F}(\Upsilon^{-1}_{a+})$$

As with the other notations, when which $A \in \mathcal{K}$ we are referring to is unclear we add a superscript A, for example $(\Lambda^{-1}_\infty)^A$.

Two further definitions relating to sets being used will now be presented. Firstly a variant of the angle between sets , and then a generalisation of the $i(n,x)$ notation.

Definition 7.9.
Let L_1, L_2 be any two stright lines in $\mathbb{R}^2$ and L^1, L^2 be the extensions of these lines to simply connected lines of infinite length in both directions. We then denote the smaller of the two types of angles that occur at the intersection of L^1 and L^2 by $\psi^{L_1}_{L_2}$.

Definition 7.10.
Let $A \in \mathcal{K}$ and $B \subset A_{0,0}$. suppose for a $n \in \mathbb{N}$, $i(n,x)$ is uniform for all $x \in B$. Then we will sometimes for convenience denote this common value $i(n,B)$.

Further notations will occasionally be used, but not regularly and so will be defined as they are used. We continue now with further definitions and properties relating to the above terms and $\mathcal{F}$ which will be necessary in our main results concerning sets in $\mathcal{K}$ which will be presented in the next chapter.

Definition 7.11.
We define, for $r \in \mathbb{R}$, the collection A^r by

$$A^r := \{A : A \textit{ is an } A_\varepsilon \textit{ type set and } \tilde{\theta}^A_{\cdot} = r\}.$$

We now state formally, to connect to the previous work, which A^r sets that our previous sets Γ_ε and A_ε are members of.

Proposition 7.5.
$\Gamma_\varepsilon \in A^{tan^{-1}(2\varepsilon)}$ and $A_\varepsilon \in A^0$.

Proof:
that $\Gamma_\varepsilon \in A^{tan^{-1}(2\varepsilon)}$ follows from the definition of Γ_ε since we can calculate from the construction that $\theta^{\Gamma_\varepsilon}_{n,\cdot} \equiv tan^{-1}(2\varepsilon)$. Since $\theta^A_{n,\cdot}$ is constant and from the proof of Lemma 5.4 $\lim_{n\to\infty} \theta^A_{n,\cdot} = 0$ it follows that $A \in A^0$. $\diamondsuit$

We now wish to investigate some of the properties possesed by $\mathcal{F}$ and resultant from the definitions that we have just made. We first look at three results concerning the $\theta_{n,i}$. We see that the stretch (and when $\mathcal{F}$ has appropriate properties the Jacobian) that occurs to each $\tilde{A}_n$ is described by a product of the base angles. Secondly we consider a convergence equivalence of this stretch factor to a convergence of the sum, which can be thought of as a test of whether a set $A \in \mathcal{K}$ spirals infinitely or not. Finally we look at the first of several results we have concerning the density of A around the image of a considered point in $A_{0,0}$.

Lemma 7.2.
For any $A \in \mathcal{K}$, $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$

$$\mathcal{H}^1(A^A_{n,i}) = \frac{\mathcal{H}^1(A^A_{0,0})}{2^n} \prod_{j=1}^{n} \frac{1}{cos(\theta^A_{j,D_{j,i}})}.$$

Proof:
By considering the right angled triangle consisting of $A^A_{n,i}$, half of the base $A^A_{n-1,j}$ of the triangular cap in which $A^A_{n,i}$ arises and the line connecting the ends that don't meet, we see that

$$\frac{\mathcal{H}^1(A^A_{n-1,j})}{2\mathcal{H}^1(A^A_{n,i})} = cos(\theta^A_{n,D_{n,i}})$$

so that

$$\mathcal{H}^1(A^A_{n,i}) = \frac{\mathcal{H}^1(A^A_{n-1,j})}{2cos(\theta^A_{n,D_{n,i}})}.$$

Thus repeating this step inductively we get

$$\begin{aligned}
\mathcal{H}^1(A^A_{n,i}) &= \frac{\mathcal{H}^1(A^A_{n-1,j})}{2cos(\theta^A_{n,D_{n,i}})} \\
&= (cos(\theta^A_{n,D_{n,i}})^{-1})(cos(\theta^A_{n-1,D_{n-1,i}}))^{-1}\frac{1}{4}\mathcal{H}^1(A^A_{n-2,\cdot}) \\
&= ... \\
&= \left(\prod_{j=0}^{n-1}(cos(\theta^A_{j,D_{j,i}}))^{-1}\right)\mathcal{H}^1(A^A_{0,1})
\end{aligned}$$

as required. $\diamondsuit$

Proposition 7.6.
Let $A \in A^0$ and $x \in A$. Then

$$\prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1} < \infty \Longleftrightarrow \sum_{n=0}^{\infty}(\theta^A_{n,i(n,x)})^2 < \infty.$$

Proof:
We first show that the claim is true for a sequence $\{\theta^A_{n,i(n,x)}\}$ composed of entirely sufficiently small elements. Where what sufficiently small entails will be shown in the proof.

Let $M := \prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1}$ and note

$$\begin{aligned}
ln(M) &= ln\left(\prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1}\right) \\
&= \sum_{n=0}^{\infty} ln((cos(\theta^A_{n,i(n,x)}))^{-1})
\end{aligned}$$

so that using a Taylor expansion for ln around 1 we have

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(cos\theta^A_{n,i(n,x)})^j}{j}\left(\frac{1}{cos\theta^A_{n,i(n,x)}}-1\right)^j \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(1-cos\theta^A_{n,i(n,x)})^j}{j}.
\end{aligned}$$

So that since

$$\lim_{x\to\infty} xsinx = 2\lim_{x\to\infty} 1-cosx$$

we have that for sufficiently small $\theta^A_{n,i(n,x)}$ that

$$(1-cos\theta^A_{n,i(n,x)}) \in (c_1\theta^A_{n,i(n,x)}sin\theta^A_{n,i(n,x)}, c_2\theta^A_n sin\theta^A_{n,i(n,x)})$$

for $0 < c_1 < c_2 < 1$ and that

$$c_2\frac{(\theta^A_{n,i(n,x)})^j sin^j\theta^A_{n,i(n,x)}}{j} < c_1\frac{1}{2}\frac{(\theta^A_{n,i(n,x)})^{j-1}sin^{j-1}\theta^A_{n,i(n,x)}}{j-1}$$

and thus that

$$\begin{aligned}
ln(M) &= \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(1-cos\theta^A_{n,i(n,x)})^j}{j} \\
&\in \left(\sum_{n=0}^{\infty} c_1\theta_{n,i(n,x)}sin\theta^A_{n,i(n,x)} - \frac{c_2(\theta^A_{n,i(n,x)})^2 sin^2\theta^A_{n,i(n,x)}}{2}, \sum_{n=0}^{\infty} c_1\theta^A_{n,i(n,x)}sin\theta^A_{n,i(n,x)}\right).
\end{aligned}$$

Since

$$\lim_{x\to\infty} x^2 = \lim_{x\to\infty} xsinx$$

we have, again for sufficiently small $\theta^A_{n,i(n,x)}$ that

$$\begin{aligned}
ln(M) &\in \left(\sum_{n=0}^{\infty} c_1\theta_{n,i(n,x)}sin\theta^A_n - \frac{c_2(\theta^A_{n,i(n,x)})^2 sin^2\theta^A_{n,i(n,x)}}{2}, \sum_{n=0}^{\infty} c_1\theta^A_{n,i(n,x)}sin\theta^A_{n,i(n,x)}\right) \\
&\in \left(\sum_{n=0}^{\infty} \frac{c_1}{2}(\theta^A_{n,i(n,x)})^2 - \frac{c_2(\theta^A_{n,i(n,x)})^2 sin^2\theta^A_{n,i(n,x)}}{2}, \sum_{n=0}^{\infty} 2c_2(\theta^A_{n,i(n,x)})^2\right) \\
&= \left(\sum_{n=0}^{\infty} c_3(\theta^A_{n,i(n,x)})^2, \sum_{n=0}^{\infty} c_2(\theta^A_{n,i(n,x)})^2\right)
\end{aligned}$$

for an appropriate $0 < c_3 < c_2$. It follows that

$$M \in \left(e^{\sum_{n=0}^{\infty} c_3(\theta^A_{n,i(n,x)})^2}, e^{\sum_{n=0}^{\infty} c_2(\theta^A_{n,i(n,x)})^2}\right)$$

and thus for $\{\theta^A_{n,i(n,x)}\}$ being comprised of sufficiently small terms we have

$$\prod_{n=0}^{\infty}(cos(\theta^A_{n,i(n,x)}))^{-1} < \infty \Longleftrightarrow \sum_{n=0}^{\infty}(\theta^A_{n,i(n,x)})^2 < \infty.$$

The general case follows from noting that since $A \in A^0$, $\theta^a_{n,i(n,x)} \to 0$ and thus for sufficiently large n the tail will be a sequence of sufficiently small $\theta^A_{n,i(n,x)}$. Since $\theta^A_{n,i(n,x)} < \pi/32$ in all cases, the finite number of terms at the begining of the sequnce will be a finite multiplying or adding factor for both sequences and thus will not affect convergence. $\diamondsuit$

We now present the first of three results that will be presented addressing the density of points in an $A \in \mathcal{K}$. The density is important as it will be the key to the existence or non-existence of approximate tangent spaces to A, and therefore an essential ingredient in discussing the rectifiability of sets in $\mathcal{K}$.

Corollary 7.1.
Let A be an A_ε type set. Then

$$\Theta^1(\mathcal{H}^1, A, y) \geq \frac{1}{2}\lim_{j\to\infty}\prod_{n=j}^{\infty}(cos\theta^A_{n,\cdot})^{-1}.$$

In particular, for A_ε type sets A such that $\prod_{n=0}^{\infty}(cos\theta^A_{n,\cdot})^{-1} = \infty$, $\Theta^1(\mathcal{H}^1, A, y) = \infty$ for all $y \in \bar{A}$.

Proof:
Let $\rho > 0$ and $y \in \bar{A}$, then there is a $j \in \mathbb{N}$ and $i \in \{1, ..., 2^j\}$ with $T^A_{j,i} \subset B_\rho(y)$ and $\mathcal{H}^1(A^A_{j,i}) \geq \rho$. From the proof of Lemma 7.2 we know

$$\mathcal{H}^1(A \cap T^A_{j,i}) = \mathcal{H}^1(A^A_{j,i})\prod_{n=j}^{\infty}(cos\theta^A_{n,\cdot})^{-1}.$$

so that

$$\mathcal{H}^1(A \cap B_\rho(y)) \geq \frac{\mathcal{H}^1(A^A_{j,i})\prod_{n=j}^{\infty}(cos\theta^A_{n,\cdot})^{-1}}{2\rho} \geq \frac{1}{2}\prod_{n=j}^{\infty}(cos\theta^A_{n,\cdot})^{-1}$$

and thus

$$\Theta^1(\mathcal{H}^1, A, y) = \lim_{\rho\to 0}\frac{\mathcal{H}^1(A \cap B_\rho(y))}{2\rho} > \frac{1}{2}\lim_{j\to\infty}\prod_{n=j}^{\infty}(cos\theta^A_{n,\cdot})^{-1}.$$

$\diamondsuit$

7.4 Properties of The Bijective Functions

We now examine some important properties of the functions F_n and the function $\mathcal{F}$. In order to work with $\mathcal{F}$ properly we must first check that it has some basic properties. We show that the function $\mathcal{F}$ is continuous and measurable. We show that images of compact sets are compact. We show that positive measure is preserves. A less well behaved, but nonetheless important property, is then to show that under conditions on Λ_∞^{-1} sets of positive measure have images of infinite measure. We additionally prove the $A \in \mathcal{K}$ version of Corollary 7.1. First of all, however, we prove that parts of the limit function $\mathcal{F}$ can be expressed as Lipschitz functions. Recalling that $\tilde{\Pi}^A_\cdot$ can be seen as the stretching (or Jacobian) factor of $\mathcal{F}$ it would seem sensible that when this is bounded, we are actually looking at a Lipschitz function. We show that this is true after defining how we make bounds. We make bounds by simply looking at the restriction of the function to pre-image sets on which $\tilde{\Pi}^A$ is bounded.

Definition 7.12.
Let $A \in \mathcal{K}$, then we define

$$\mathcal{F}_m := \mathcal{F}|_{\Lambda_m^{-1}}.$$

Lemma 7.3.
For $m \in \mathbb{R}$, $\mathcal{F}_m := \mathcal{F}|_{\Lambda_m^{-1}}$ is Lipschitz with $Lip\mathcal{F}_m \leq Cm^2$.

Proof:
Let $x, y \in \Lambda_m^{-1}$, and without loss of generality let $y < x$. there are then two cases to consider

1. $\{ty + (1-t)x : t \in [0,1]\} \subset \Lambda_m^{-1}$,
2. otherwise.

Case 1 is the simpler. In this case we have $\mathcal{F}_m|_{[y,x]} = \mathcal{F}|_{[y,x]}$ and $\tilde{\Pi}(z) \in [y, x]$. It follows from the construction of the F_n from which $\mathcal{F}$ is defined as a limit that

$$d(F_n(y), F_n(x)) \leq md(y, x), \text{ for all } n \in \mathbb{N},$$

which impples

$$\begin{aligned} d(\mathcal{F}(y), \mathcal{F}(x)) &\leq \limsup_{n\to\infty} d(F_n(y), F_n(x)) \\ &\leq \limsup_{n\to\infty} md(y, x) \\ &= md(y, x). \end{aligned}$$

For case 2 we know that there must exist a $z \in (y,x)$ such that $\tilde{\Pi}(x) > m$ and therefore there is an $n_0 \in \mathbb{N}$ such that

$$y, x \notin T^A_{n_0, i(n_0,z)}$$

and indeed

$$i(n_0, y) < i(n_0, z) < i(n_0, x).$$

It follows that we can find a minimum such n_0 and therefore an $n_1 \in \mathbb{N}$ such that

$$i(n_1, x) - 3 \leq i(n_1, y) \leq i(n_1, x) - 2$$

and such that for all $n < n_1$

$$i(n, y) \in \{i(n, x) - 1, i(n, x).$$

From this, it follows firstly that for each $n < n_1$

$$[y, x] \subset T^A_{n,i(n,y)} \cup T^A_{n,i(n,x)}$$

which implies that $F_{n_0}|_{[y,x]}$ has Lipschitz constant

$$Lip F_{n_0}|_{[y,x]} \leq \max\{\tilde{\Pi}^A_{n_0}(x), \tilde{\Pi}^A_{n_0}(y)\} \leq m$$

so that

$$d(F_{n_0}(y), F_{n_0}(x)) \leq m d(y, x).$$

It also follows from the choice of n_1 that

$$d(\mathcal{F}(y), \mathcal{F}(x)) < 2 \max_{w \in \{x,y\}} \mathcal{H}^1(A^A_{n_0-1, i(n_0-1,w)}).$$

Now, using Lemma 5.1 we know

$$\pi_x(O_{A^A_{n_0,i(n_0,y)}}(T^A_{n_0,i(n_0,y)} \cap T^A_{n_0,i(n_0,x)})) \cap O_{A^A_{n_0,i(n_0,y)}}(A^A_{n_0,i(n_0,y)}) = \emptyset$$

and thus

$$\begin{aligned} d(F_{n_0}(y), F_{n_0}(x)) &\geq \mathcal{H}^1(A^A_{n_0,i(n_0,y)+1}) \\ &\geq \frac{1}{2} \min_{w \in y,x} \mathcal{H}^1(A^A_{n_0-1,i_w(n_0-1)}) \end{aligned}$$

the latter following since $A^A_{n_0,i(n_0,y)+1}$ is a shorter side of either $A^A_{n_0-1,i(n_0-1,y)}$ or $A^A_{n_0-1,i(n_0-1,x)}$.

Since

$$y \in T^A_{n_0,i(n_0,y)+1} \Rightarrow 1 \leq \tilde{\Pi}^A_{n_0-1}(y) \leq m$$

and

$$x \in T^A_{n_0,i(n_0,x)+1} \Rightarrow 1 \leq \tilde{\Pi}^A_{n_0-1}(x) \leq m$$

it follows that

$$\max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}) \leq m \min_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}).$$

Thus

$$d(F_{n_0}(y), F_{n_0}(x)) \geq \frac{1}{2m} \max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}).$$

Hence

$$\begin{aligned} d(\mathcal{F}(y), \mathcal{F}(x)) &\leq 2 \max_{w\in\{x,y\}} \mathcal{H}^1(A^A_{n_0-1,i(n_0-1,w)}) \\ &\leq 4md(F_{n_0}(y), F_{n_0}(x)) \\ &\leq 4m^2 d(y,x). \end{aligned}$$

Combining the two cases gives us, using $m \geq 1$

$$\begin{aligned} d(\mathcal{F}(y), \mathcal{F}(x)) &\leq \max\{m, 4m^2\} d(x,y) \\ &= 4m^2 d(x,y) \end{aligned}$$

for each $x, y \in \Lambda^{-1}_m$. ♢

Proposition 7.7.
Let $A \in \mathcal{K}$ and let $\mathcal{F}$ be the function related to A. Then

1. *$\mathcal{F}$ is continuous,*
2. *should $B \subseteq A_0$ be closed, then $\mathcal{F}(B) \subseteq A$ is compact,*
3. *if $B \subset A_0$ is such that $\mathcal{H}^1(B) > 0$ then $\mathcal{H}^1(\mathcal{F}(B)) > \mathcal{H}^1(B)/6 > 0$,*
4. *if $\mathcal{H}^1(\Lambda^{-1}_\infty) > 0$ then $\mathcal{H}^1(\Lambda_\infty) = \infty$, and*
5. *if $\Theta^1(x, \mathcal{H}^1, \Lambda^{-1}_\infty) > 0$ then $\Theta^1(\mathcal{F}(x), \mathcal{H}^1, \Lambda_\infty) = \infty$*
6. *$\mathcal{F}$ is $\mathcal{H}^1$-measurable.*

Proof:
As we are considering only one A we shall omit the A superscripts.

For (1),

since for all constructions A that we consider we have $\theta_{0,0} \leq \pi/32$ we see that

$$diam(T_{n,\cdot}) = \mathcal{H}^1(A_{n,\cdot}) \leq \frac{(cos(\pi/32))^{-1}}{2}\mathcal{H}^1(A_{n-1,\cdot})$$

which inductively gives us

$$diam(T_{n,i}) \leq \left(\frac{(cos(\pi/32))^{-1}}{2}\right)^n \mathcal{H}^1(A_0).$$

Since $cos(\pi/32) > 1/2$, $(cos(\pi/32))^{-1}/2 < 1$ so that

$$\lim_{n\to\infty} diam(T_{n,\cdot}) = 0.$$

It follows that for all $\varepsilon > 0$, $diam(T_{n,\cdot}) < \varepsilon/2$ for all n greater than some sufficiently large n_0. Consider $x_1, x_2 \in A_0$ such that $|x_1 - x_2| < 2^{-n_0}$. Then

$$x_1, x_2 \in \left[\frac{i-1}{2^n_0}, \frac{i}{2^n}\right] \cup \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]$$

for some $i \in \{1, ..., 2^n - 1\}$, so that since $\mathcal{F}([(i-1)2^{-n}, i2^{-n}]) \subset T_{n,i}$ for each $n \in \mathbb{N}$ and each $i \in \{1, ..., 2^n\}$

$$\mathcal{F}(x_1), \mathcal{F}(x_2) \in T_{n_0,i} \cup T_{n_0,i+1},$$

which implies

$$|\mathcal{F}(x_1) - \mathcal{F}(x_2)| \leq diam(T_{n_0,i-1}) + diam(T_{n_0,i}) < \varepsilon.$$

For (2),
Since A_0 is bounded, so to is any closed subset of A_0, thus should B be a closed subset of A_0 it is also compact. It then follows from the fact that $\mathcal{F}$ is continuous that $\mathcal{F}(B)$ is closed and indeed bounded since $\mathcal{F}(A_0) \subset [0,1] \times [0,1]$ and thus also compact.

For (3)
Let our set, for convenience be denoted K. Let $\mathcal{H}^1(K) > 0$, say $\mathcal{H}^1(K) =: \beta$. It follows that there is a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$

$$\mathcal{H}^1_\delta(K) > \frac{\beta}{2}.$$

Now, let $\delta < \delta_0$ and $\{B_\delta\}$ be a δ-cover of $\mathcal{F}(K)$ and consider a $B \in B_\delta$. By Lemma 5.1 we see that there is an $n(B) \in \mathbb{N}$ such that whether or not

$center(B) \in \mathcal{F}(K)$ $B \cap \mathcal{F}(B)$ subset $T^A_{n(B),i(B)-1} \cup T^A_{n(B),i(B)} \cup T^A_{n(B),i(B)+1}$ for some ß$(B) \in \{2, ..., 2^{n(B)} - 1\}$with

$$diam(T^A_{n(B),\cdot}) = length(A^A_{n(B),\cdot}) \in \left(\frac{diam(B)}{2}, diam(B)\right)$$

so that

$$diam(B) > \sum_{j=i(B)-1}^{i(B)+1} diam(T^A_{n(B),\cdot}).$$

In this case we also have

$$\begin{aligned} \mathcal{F}(B \cap \mathcal{F}(K)) \quad &\subset \quad \bigcup_{j=i(B)-1}^{i(B)+1} \mathcal{F}^{-1}(T_{n(b),j}) \\ &= \quad \bigcup_{j=1(B)-1}^{i(B)+1} F^{-1}_{n(b)}(A_{n(B),j}) \end{aligned}$$

which, since $F_{n(B)}$ is an expansion map, gives three intervals $I_{B,j}$, $j = 1, 2, 3$ with

$$diam(I_{B,j}) = length(I_{B,j}) \leq length(A_{n(B),j}) < diam(B) < \delta.$$

It follows that

$$\sum_{j=i(B)-1}^{i(B)+1} diam(I_{B,j}) < 3diam(B).$$

Since

$$\mathcal{F}(K) \subset \bigcup_{B \in B_\delta} (B \cap \mathcal{F}(K))$$

it follows that

$$K \subset \bigcup_{B \in B_\delta} \bigcup_{j=i(B)-1}^{i(B)+1} I_{B,j}$$

which implies that $\{\{I_{B,j}\}_{B \in B_\delta}\}_{j=i(B)-1}^{i(b)+1}$ is a δ cover of K and thus that

$$\sum_{B \in B_\delta} \sum_{j=i(B)-1}^{i(B)+1} I_{B,j} \geq \mathcal{H}^1_\delta(K) > \frac{\beta}{2}$$

and therefore

$$\sum_{B \in B_\delta} diam(B) > \frac{1}{3} \sum_{B \in B_\delta} \sum_{j=i(B)-1}^{i(B)+1} I_{B,j} > \frac{1}{3}\frac{\beta}{2} = \frac{\beta}{6}.$$

Since this is true for any such δ-cover of $\mathcal{F}(K)$ we see that

$$\mathcal{H}^1_\delta(\mathcal{F}(K)) > \frac{\beta}{6}$$

for any $\delta < \delta_0$ and therefore that

$$\begin{aligned}\mathcal{H}^1(\mathcal{F}(K)) &= \lim_{\delta\to 0}\mathcal{H}^1_\delta(\mathcal{F}(K)) \\ &\geq \lim_{\delta\to 0}\frac{\beta}{6} \\ &= \frac{\beta}{6} \\ &> 0. \end{aligned} \tag{7.1}$$

For (4),
Let $M > 0$. Then since $\mathcal{H}^1_{A_0}$ is Radon and

$$\Lambda^{-1}_\infty = \bigcup_{n\in\mathbb{N}}\{x \in \Lambda^{-1}_\infty : \tilde{\Pi}_n(x) > M\}$$

it follows that there is an $n_0 \in \mathbb{N}$ with

$$\mathcal{H}^1(\{x \in \Lambda^{-1}_\infty : \tilde{\Pi}_{n_0}(x) > M\}) > \frac{\mathcal{H}^1(\Lambda^{-1}_\infty)}{2} > 0.$$

We set

$$\Lambda^{-1}_{\infty,n_0} := \{x \in \Lambda^{-1}_\infty : \tilde{\Pi}_{n_0}(x) > M\}.$$

It follows that with

$$X := \{i \in \{1, ..., 2^{n_0}\} : T_{n_0,i} \cap F_{n_0}(\Lambda^{-1}_{\infty,n_0} \neq \emptyset\}$$

$$\begin{aligned}\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0})) &= \sum_{i\in X}\mathcal{H}^1(F_{n_0}(\Lambda^{-1}_{\infty,n_0}) \cap T_{n_0,i}) \\ &> M\sum_{i\in X}\mathcal{H}^1(\Lambda^{-1}_{\infty,n_0} \cap [2^{-n_0}(i-1), 2^{n_0}i]) \\ &= M\mathcal{H}^1(\Lambda^{-1}_{\infty,n_0}) \\ &> M\frac{\mathcal{H}^1(\Lambda^{-1}_\infty)}{2}.\end{aligned}$$

We then apply (3) to each set $A^{n_0,i} \in \mathcal{K}$ defined as the subconstruction (and subset) of A starting with $A_0^{n_0,i} = A_{n_0,i}$ to find

$$\mathcal{H}^1(\mathcal{F} \circ F_{n_0}^{-1}(F_{n_0}(\Lambda_{\infty,n_0}^{-1}) \cap T_{n_0,i})) > \frac{\mathcal{H}^1(F_{n_0}(\Lambda_{\infty,n_0}^{-1}) \cap T_{n_0,i})}{6}$$

and thus that

$$\begin{aligned}
\mathcal{H}^1(\mathcal{F}(\Lambda_{\infty,n_0}^{-1})) &= \sum_{i \in X} \mathcal{H}^1(\mathcal{F} \circ F_{n_0}^{-1}(F_{n_0}(\Lambda_{\infty,n_0}^{-1}) \cap T_{n_0,i})) \\
&> \frac{1}{6} \sum_{i \in X} \mathcal{H}^1(F_{n_0}(\Lambda_{\infty,n_0}^{-1}) \cap T_{n_0,i}) \\
&= \frac{1}{6} \mathcal{H}^1(F_{n_0}(\Lambda_{\infty,n_0}^{-1})).
\end{aligned}$$

We therefore now have

$$\mathcal{H}^1(\mathcal{F}(\Lambda_{\infty,n_0}^{-1})) > \frac{1}{6} \mathcal{H}^1(F_{n_0}(\Lambda_{\infty,n_0}^{-1})) > \frac{M \mathcal{H}^1(\Lambda_\infty^{-1})}{12}.$$

Since this is true for each $M > 0$ it follows that

$$\mathcal{H}^1(\mathcal{F}(\Lambda_\infty^{-1})) > \mathcal{H}^1(\mathcal{F}(\Lambda_{\infty,n_0}^{-1})) = \infty.$$

For (5),
Suppose $x \in \Lambda_\infty^{-1}$ is such that $\Theta^1(x, \mathcal{H}^1, \Lambda_\infty^{-1}) > 0$.

Consider $\mathcal{F}(x)$ and let $\rho > 0$. We know firstly from definition that there is an $n_0 > 0$ such that $F_n(x) \in B_{\rho/2}(\mathcal{F}(x))$ for all $n > n_0$ and thus, since from the proof of (1) $diam(T_{n,i}) \to 0$ as $n \to \infty$, there is an $n_1 \geq n_0$ such that $diam(T_{n_1,\cdot}) < \rho/4$ and thus $\cup_{j=-1}^{1} T_{n_1, i(x,n_1)+j} \subset B_\rho(\mathcal{F}(x))$.

For the remainder of (5) we write $i := i(n_1, x)$ We now, temporarily have two cases to consider, namely CASE I that $F_n(x) \in E(T_{n_1,i})$ and CASE II that $F_n(x) \in T_{n_1,i} - E(T_{n_1,i})$.

CASE I:
In this case $F_{n_1}(x) \in T_{n_1,i} \cap T_{n_1,i-1}$ or $F_{n_1}(x) \in T_{n_1,i} \cap T_{n_1,i+1}$, without loss of generality let us suppose that it is the latter case. Then

$$\begin{aligned}
x &= i2^{-n_1} \\
&\in ((i-1)2^{-n_1}, (i+1)2^{-n_1}) \\
&\subset [(i-1)2^{-n_1}, (i+1)2^{-n_1}] \\
&= F_n^{-1}(A_{n,i} \cup A_{n,i+1}).
\end{aligned}$$

Since $\Theta^1(x, \mathcal{H}^1, \Lambda_\infty^{-1}) > 0$ it follows that

$$\mathcal{H}^1(\Lambda_\infty^{-1} \cap [(j-1)2^{-n_1}, j2^{-n_1}] > 0$$

for atleast one $j \in \{i, i+1\}$. Without loss of generality let us assume that $j = i$. Then

$$\begin{aligned}
\mathcal{H}^1(F_{n_1}^{-1}(\Lambda_\infty^{-1}) \cap A_{n_1,i}) &= \tilde{\Pi}_{n_1}(x)\mathcal{H}^1(\Lambda_\infty^{-1} \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&\geq \mathcal{H}^1(\Lambda_\infty^{-1} \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&> 0.
\end{aligned}$$

CASE II:
In this case $F_{n_1}(x) \in T_{n_1,i} - E(T_{n_1,i})$ so that

$$x \in ((i-1)2^{-n_1}, i2^{-n_1}) \subset [(i-1)2^{-n_1}, i2^{-n_1}] = F_{n_1}^{-1}(A_{n,i}).$$

Thus since $\Theta^1(x, \mathcal{H}^1, \Lambda_\infty^{-1}) > 0$ it follows that

$$\mathcal{H}^1(\Lambda_\infty^{-1} \cap [(i-1)2^{-n_1}, i2^{-n_1}] > 0$$

and therefore

$$\begin{aligned}
\mathcal{H}^1(F_{n_1}^{-1}(\Lambda_\infty^{-1}) \cap A_{n_1,i}) &= \tilde{\Pi}_{n_1}(x)\mathcal{H}^1(\Lambda_\infty^{-1} \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&\geq \mathcal{H}^1(\Lambda_\infty^{-1} \cap [(i-1)2^{-n_1}, i2^{-n_1}]) \\
&> 0.
\end{aligned}$$

That is, in either case there is a $n \in \mathbb{N}$ and $i \in \{1, ..., 2^n\}$ such that $T_{n,i} \subset B_\rho(\mathcal{F}(x))$ and $\mathcal{H}^1(F_n^{-1}(\Lambda_\infty^{-1}) > 0$. Applying (iv) to the $A_1 \in \mathcal{K}$ resulting from the subconstruction of A on $T_{n,i}$ it follows that $\mathcal{H}^1(\Lambda_\infty \cap T_{n,i}) = \infty$ and thus that $\mathcal{H}^1(B_\rho(\mathcal{F}(x)) \cap \Lambda_\infty) = \infty$. Since this is true for all $\rho > 0$ it follows that

$$\Theta^1(\mathcal{F}(x), \mathcal{H}^1, \Lambda_\infty) = \infty,$$

completing the proof of (5).

Proof of (6):
We note that the open sets of A with respect to $\mathcal{H}^1$ measure are $U \cap A$ for U open in the usual sense in $\mathbb{R}^2$. Now consider an open set in A, $V := U \cap A$ for some U open in $\mathbb{R}^2$.

Let

$$\mathcal{T}_1 := \cup\{T_{1,i} : T_{1,i} \subset U\}$$

and in general

$$\mathcal{T}_n := \cup\{T_{n,i} : T_{n,i} \subset U\}.$$

We claim that

$$V = \bigcup_{n=1}^{\infty}(\mathcal{T}_n \cap A).$$

Clearly $\mathcal{T}_n \cap A \subset U \cap A$ for all $n \in \mathbb{N}$ and thus

$$\bigcup_{n=1}^{\infty}(\mathcal{T}_n \cap A) \subset U \cap A = V.$$

Conversely, let $x \in V$. Then $x \in A$ and there exists $\rho > 0$ such that $B_\rho(x) \subset U$. Since we know that for any $A \in \mathcal{K}$, and $x \in A$

$$\lim_{n\to\infty} diam(T_{n,i(n,x)}) = 0$$

there exists $n_\rho \in \mathbb{N}$ such that $diam(T_{n_\rho,i(n_\rho,x)} < \rho/2$. Then

$$T_{n_\rho,i(n_\rho,x)} \subset B_\rho(x) \subset U$$

thus

$$T_{n_\rho,i(n_\rho,x)} \subset \mathcal{T}_{n_\rho}$$

and thus $x \in \mathcal{T}_{n_\rho}$.

Since $x \in A$ we have $x \in \mathcal{T}_{n_\rho} \cap A$ and thus

$$x \in \bigcup_{n=1}^{\infty}(\mathcal{T}_{n_{rho}} \cap A).$$

It follows that

$$V \subset \bigcup_{n=1}^{\infty}(\mathcal{T}_{n_{rho}} \cap A).$$

Now, for each $n \in \mathbb{N}$

$$\mathcal{T}_n \cap A = \bigcup_{i\in I_n} T_{n,i} \cap A$$

for some (possibly empty) index $I_n \subset \{0, 1, ..., 2^n - 1\}$. Thus

$$\mathcal{F}^{-1}(\mathcal{T}_n \cap A) = \bigcup_{i\in I_n} D_{n,i}$$

where $D_{n,i}$ is the i-th dyadic interval of order n. Thus

$$\begin{aligned}\mathcal{F}^{-1}(V) &= \mathcal{F}^{-1}\left(\bigcup_{n=1}^{\infty}(\mathcal{T}_n \cap A)\right)\\ &= \bigcup_{n=1}^{\infty}\mathcal{F}^{-1}(\mathcal{T}_n \cap A)\\ &= \bigcup_{n=1}^{\infty}\bigcup_{i\in I_n} D_{n,i}\end{aligned}$$

which is a Borel set in $A_{0,0}$ and thus $\mathcal{H}^1$-measurable. It follows that for any Borel set $B \in A$ $\mathcal{F}^{-1}(B)$ is a Borel set in $A_{0,0}$. Thus, finally, if B is a $\mathcal{H}^1$-measurable set in A, $\mathcal{F}^{-1}(B)$ is a $\mathcal{H}^1$-measurable set in $A_{0,0}$. The fact that the measurability of the inverse images of measurable sets follows from the measurability of the inverse images of open sets is standard measure theory and is discussed in, for example, Rudin [14] or Bartle [2]. $\diamondsuit$

To complete the preliminary results required for our study of measure and rectifiability of sets in $\mathcal{K}$ we have one more lemma concerning density to consider. It is this final general density Lemma that will be applied in the proof of non-rectifiability of those Koch sets which are not rectifiable (which ones they are will be made clear later). It shows the presence of infinite density almost everywhere in the image of any measurable subset of Λ_∞^{-1} of positive measure. In order to prove this Lemma, however, we first need a couple of general measure theoretic results showing that the set of points density one are sufficiently large in a set of positive measure in $A_{0,0}$. The second is a condition of non-rectifiability.

Proposition 7.8.
Let $B \subset A_0$ be $\mathcal{H}^1$-measurable, then

$$\mathcal{H}^1(\{x \in B : \Theta^1(x, \mathcal{H}^1, B) = 1\}) = \mathcal{H}^1(B).$$

Proof:
Since B is $\mathcal{H}^1$-measurable we know that for all $\rho > 0$

$$1 = (2\rho)^{-1}\mathcal{H}^1(B_\rho(x)) = (2\rho)^{-1}(\mathcal{H}^1(B_\rho(x)\cap B) + \mathcal{H}^1(B_\rho(x)\cap B^c))$$

so that

$$\begin{aligned}1 &= \lim_{\rho\to 0}(2\rho)^{-1}\mathcal{H}^1(B_\rho(x))\\ &= \lim_{\rho\to 0}(2\rho)^{-1}(\mathcal{H}^1(B_\rho(x)\cap B) + \mathcal{H}^1(B_\rho(x)\cap B^c))\\ &= \Theta^1(x, \mathcal{H}^1, B) + \Theta^1(x, \mathcal{H}^1, B^c).\end{aligned}$$

From standard theory (see for example [Simon3] Theorem 3.5) we know $\Theta^1(x, \mathcal{H}^1, C) = 0$ for $\mathcal{H}^1$-almost all $x \in C^c$ for any $\mathcal{H}^1$-measurable set C with $\mathcal{H}^1(C) < \infty$. Hence $\Theta^1(x, \mathcal{H}^1, B^c) = 0$ for $\mathcal{H}^1$-almost all $x \in B$ and thus

$$\Theta^1(x, \mathcal{H}^1, B) = 1 - \Theta^1(x, \mathcal{H}^1, B^c) = 1$$

for almost all $x \in B$. The result follows ♢

Proposition 7.9.
Let $A \subset \mathbb{R}^2$. Let θ be a $L^1(\mathcal{H}^1, \mathbb{R}^2, \mathbb{R})$ positive function on A. Suppose that B is a subset of A of positive measure that satisfies $\theta(x) \geq r > 0$ for all $x \in B$. Let $x \in B$ satisfy

$$\Theta^1(\mathcal{H}^1, A, x) \geq \Theta(\mathcal{H}^1, B, x) = \infty.$$

Then A does not have a 1-dimensional approximate tangent plane for A at x with respect to θ.

Proof:
Let P be any potential approximate tangent plane for A at x with respect to θ and define $\phi \in C_c^0(\mathbb{R}^2; \mathbb{R})$ by

$$\phi(x) := \begin{cases} 1 & |x| \leq 1 \\ 2 - |x| & 1 \leq |x| \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

We then have

$$\int_P \phi d\mathcal{H}^1 = 3.$$

However,

$$\begin{aligned} \lim_{n\to\infty} \lambda^{-1} \int_A \phi(\lambda^{-1}(z-x)) d\mathcal{H}^1(z) &\geq \lim_{n\to\infty} \lambda^{-1} \int_B \phi(\lambda^{-1}(z-x)) d\mathcal{H}^1(z) \\ &\geq \lim_{n\to\infty} \lambda^{-1} \int_{B\cap B_\lambda(x)} r d\mathcal{H}^1 \\ &= r \lim_{n\to\infty} \lambda^{-1} \int_{B\cap B_\lambda(x)} 1 d\mathcal{H}^1 \\ &= r \lim_{\lambda\to 0} \frac{\mathcal{H}^1(B \cap B_\lambda(x))}{\lambda} \\ &= 2r\Theta^1(x, \mathcal{H}^1, A) \\ &> 3. \end{aligned}$$

It is therefore impossible that A have an approximatye tangent plane at x with respect to θ. ♢

Lemma 7.4.
Let $A \in \mathcal{K}$ and $\mathcal{H}^1(B \cap \Lambda_\infty^{-1}) > 0$ for some measurable subset $B \subset A_{0,0}$. Then there exists

$$B_1 \subset B \cap \Lambda_\infty^{-1},$$
$$\mathcal{H}^1(B_1) = \mathcal{H}^1(B \cap \Lambda_\infty^{-1})$$

such that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), \mathcal{F}(x)) = \infty$$

for all $x \in B_1$.

In particular, if $A \in \mathcal{K}$ and $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0$, then for $\mathcal{H}^1$-a.e. $x \in \Lambda_\infty^{-1}$

$$\Theta^1(\mathcal{H}^1, A, \mathcal{F}(x)) \geq \Theta^1(\mathcal{H}^1, \Lambda_\infty, \mathcal{F}(x)) = \infty.$$

Proof:
We note from Proposition 7.8 that

$$\Theta^1(\mathcal{H}^1, B \cap \Lambda_\infty^{-1}, x) = 1$$

for $\mathcal{H}^1$-a.e. $x \in B \cap \Lambda_\infty^{-1}$. We thus choose

$$B_1 := \{x \in B \cap \Lambda_\infty^{-1} : \Theta^1(\mathcal{H}^1, B \cap \Lambda_\infty^{-1}, x) = 1\},$$

noting that $\mathcal{H}^1(B_1) = \mathcal{H}^1(B \cap \Lambda_\infty^{-1})$ as required.

Choose now $y \in B_1$ arbitrarily.

We then note that from the definition of Θ^1 there must exist an $r_0 > 0$ so that for all $r \leq r_0$

$$(2r)^{-1}\mathcal{H}^1(B_r(y) \cap B_1) > 7/8.$$

We now claim that for any dyadic interval $D \ni y$ with $|D| := \mathcal{H}^1(D) < r_0/2$

$$\mathcal{H}^1(D \cap B_1) > 3/4|D|.$$

We see this by selecting

$$\gamma := \max\{d(y,z) : z \in E(D)\}$$

(where $E(D)$ as elsewhere denotes the endpoints of D). Then $\gamma < |D| < r_0$ and $D \subset \overline{B_\gamma(y)}$. Thus

$$\frac{\mathcal{H}^1(B_1^c \cap B_\gamma(y))}{2\gamma} = \frac{1 - \mathcal{H}^1(B_1 \cap B_\gamma(y))}{2\gamma} < \frac{1}{8}$$

which implies

$$\frac{\mathcal{H}^1(B_1^c \cap D)}{|D|} \leq \frac{2\mathcal{H}^1(B_1^c \cap B_\gamma(y))}{2\gamma} < \frac{1}{4}$$

and thus

$$\begin{aligned}\frac{\mathcal{H}^1(B_1 \cap D)}{|D|} &= \frac{|D| - \mathcal{H}^1(B_1^c \cap D)}{|D|} \\ &> (|D| - |D|/4)|D|^{-1} \\ &= \frac{3}{4},\end{aligned}$$

proving the claim.

In particular, the claim holds for any dyadic interval $D_m \ni y$ of order $m \geq m_0$ where m_0 is chosen such that $2^{-m_0} \leq r_0$.

Then, selecting, independently from one another, $1 > p > 0$ and $M \in \mathbb{R}$ with

$$\mathcal{H}^1(A_{m_0,i(y,m_0)}) = diam(T_{m_0,i(y,m_0)}) > p.$$

We choose $m \geq m_0$ such that $diam(T_{m,i(y,m)}) \in (p/2, 2p)$ and $T_{m,i(y,m)} \subset B_p(y)$. Note that $F_m^{-1}(A_{m,i(y,m)}) = A \cap T_{m,i(y,m)}$. That is, defining $\mathcal{B}_1 := \mathcal{F}(B_1)$

$$\mathcal{H}^1(B_p(y) \cap \mathcal{B}_1) \geq \mathcal{H}^1(\mathcal{B}_1 \cap T_{m,i(y,m)}) = \mathcal{H}^1(\mathcal{F}(D_m) \cap \mathcal{B}_1).$$

Since, for all $x \in B_1 \cap D_m$, $\prod_{m=0}^{\infty}(cos\theta_{n,i(y,n)})^{-1} = \infty$ there exists a $q_0 \in \mathbb{N}$ such that for

$$B^q := \left\{ x \in B_1 \cap D_m : \prod_{n=m+1}^{p} (cos\theta_{n,i(y,n)})^{-1} > M \right\}$$

$$\mathcal{H}^1(B_{q_0}) > \mathcal{H}^1(B_1 \cap D_m)/2.$$

If this were not true then since $B^q \subset B^{q+1}$ for each q it would follow that

$$\mathcal{H}^1\left(\bigcup_{q=m+1}^{\infty} B^q\right) \leq \frac{\mathcal{H}^1(B_1 \cap D_m)}{2}$$

and thus there would exist $x \in B_1 \cap D_m$ such that $\prod_{n=m+1}^{\infty}(cos\theta_{n,i(x,n)})^{-1} < M < \infty$. This contradiction confirms our claim.

We then note

$$\mathcal{H}^1(B_1) > \frac{1}{2}\mathcal{H}^1(B_1 \cap D_m) > \frac{3}{8}|D_m|$$

and that since for all $x \in D_m$, for all $x \in B_1$

$$\prod_{n=0}^{\infty}(cos\theta_{n,i(x,n)})^{-1} \geq \prod_{n=0}^{m}(cos\theta_{n,i(x,n)})^{-1} > \frac{p}{|D_m|}.$$

It then follows that for $\tilde{y} := \mathcal{F}(y)$

$$\begin{aligned}
\mathcal{H}^1(\mathcal{B}_1 \cap B_p(\tilde{y})) &\geq \mathcal{H}^1(\mathcal{B}_1 \cap T_{m,i(y,m)}) \\
&\geq \mathcal{H}^1\left(\mathcal{B}_1 \cap \bigcup_{D_q \cap B_1 \neq \emptyset} T_{q,i(D_q,q)}\right) \\
&= \sum_{D_q \cap B_1 \neq \emptyset} \mathcal{H}^1(\mathcal{B}_1 \cap T_{q,i(D_q,q)}) \\
&\geq \sum_{D_q \cap B_1 \neq \emptyset} \mathcal{H}^1(F_q(D_q \cap B_1)) \\
&= \sum_{D_q \cap B_1 \neq \emptyset} \prod_{n=0}^{q}(cos\theta_{q,i(D_q,q)})^{-1}\mathcal{H}^1(D_q \cap B_1) \\
&> \frac{p}{|D_m|} \sum_{D_q \cap B_1 \neq \emptyset} \prod_{n=m+1}^{q}(cos\theta_{q,i(D_q,q)})^{-1}\mathcal{H}^1(D_q \cap B_1) \\
&> \frac{Mp}{|D_m|} \sum_{D_q \cap B_1 \neq \emptyset} \mathcal{H}^1(D_q \cap B_1) \\
&= \frac{Mp}{|D_m|}\mathcal{H}^1\left(\bigcup_{D_q \cap B_1 \neq \emptyset} D_q \cap B_1\right) \\
&= \frac{Mp}{|D_m|}\mathcal{H}^1(B_1) \\
&> \frac{3Mp}{8|D_m|}|D_m| \\
&= \frac{3Mp}{8}
\end{aligned}$$

Since this is true for any $p < diam(T_{m_0,i(y,m_0)}$

$$\Theta^1(\mathcal{H}^1, \mathcal{B}_1, \tilde{y}) = \lim_{p \searrow 0} \frac{\mathcal{H}^1(\mathcal{B}_1 \cap B_p(\tilde{y}))}{2p} \geq \frac{3M}{16}.$$

Since this is true for each $M \in \mathbb{R}$ we have

$$\Theta^1(\mathcal{H}^1, \mathcal{B}_1, \tilde{y}) = \infty.$$

As this is true for any $y \in B_1$ it follows that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), \mathcal{F}y) = \infty$$

for each $y \in B_1$ completing the first part of the proof.

For the final part of the proof we note that $A_{0,0}$ is itself measurable and that $A_{0,0} \cap \Lambda_\infty^{-1} = \Lambda_\infty^{-1}$. It follows from the above that there is a set $B \subset \Lambda_\infty^{-1}$ with $\mathcal{H}^1(B) = \mathcal{H}^1(\Lambda_\infty^{-1}$ so that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B), \mathcal{F}(x)) = \infty$$

for all $x \in B$. Since $A_{0,0} \supset \Lambda_\infty^{-1} \supset B$, $\mathcal{F}(A_{0,0} = A$, $\mathcal{F}(\Lambda_\infty^{-1} = \Lambda_\infty$ and $\mathcal{H}^1$-a.e. $x \in \Lambda_\infty^{-1}$, $x \in B$ it follows that for all $x \in B$ and thus $\mathcal{H}^1$-a.e. in Λ_∞^{-1}

$$\Theta^1(\mathcal{H}^1, A, \mathcal{F}(x)) \geq \Theta^1(\mathcal{H}^1, \Lambda_\infty, \mathcal{F}(x)) \geq \Theta^1(\mathcal{H}^1, \mathcal{F}(B), \mathcal{F}(x)) = \infty,$$

which completes the proof. $\diamondsuit$

This completes the preliminary results that we need for the rectifiability and measure results on sets in $\mathcal{K}$.

7.5 Relative Centralisation of Semi-Self-Similar Sets

We now look at some preliminary results that we will need for results on dimension. We will reduce all of our questions to an application of the results of Hutchinsion [10] to get our dimension results. We do this, in essence, by a comparison principle. We show that sets in $\mathcal{K}$ depending on properties of $\tilde{\theta}^A$ can be dimension invariantly rearranged so that they are supersets of some sets to which Hutchinsons results apply and subsets of others. By considering sequences of such rearrangements we can deduce the dimension of our sets from the dimensions of the sets to which we are comparing.

It is infact true that we could, in principal, apply Hutchinsons results directly. However, the parameters of the sets and "self-similarity" functions cannot be (at least not easily) extracted from sets in $\mathcal{K}$. Thus actually giving an explicit dimension directly is not possible.

Our comparison principle, or rearrangement involves seperating each triangular cap in a particular approximation to some $A \in \mathcal{K}$, T_n and moving

each seperately by an orthogonal transformation in such a way that each of the newly positioned triangular caps remain disjoint. We do this by placing each inside of a triangular cap of another, larger, T_n from some other $A' \in \mathcal{K}$. Since all Hausdorff measures are translation invariant it follows that Hausdorff dimension is also translation invariant and thus the union of the replaced triangular caps is the same dimension as the original caps. We can in this way compare the dimesion of A to that of each $T_n^{A'}$ and thus of A'. It will be by selecting appropriate A' that we will prove our dimension results.

We start by defining the transformation process, which, due to the placing of one set into parts of another, we call centering. That is one set is centered in the bigger one.

Definition 7.13.
Let $A_1, A_2 \subset \mathbb{R}^2$. We say that we can center A_1 in A_2 (or that A_1 can be centered in A_2) written $A_1 \hookrightarrow^c A_2$ if for each $m \in \mathbb{N}$ there exists sets A_{1m} and A_{2m} such that

$$\bigcap_{m=1}^{\infty} A_{2m} \subset A_2, \quad A_{2m} \subset A_{2(m-1)} \quad \textit{for all} \quad m \in \mathbb{N}$$

$$A_1 \subset \bigcap_{m=1}^{\infty} A_{1m}, \quad A_{1m} \subset A_{1(m-1)} \quad \textit{for all} \quad m \in \mathbb{N};$$

that for each $m \in \mathbb{N}$ there exists $n_1(m), n_2(m) \in \mathbb{N}$, $n_1(m) \leq n_2(m)$, disjoint sets $\{A_{1mj}\}_{j=1}^{n_1(m)}$ and disjoint sets $\{A_{2mj}\}_{j=1}^{n_2(m)}$ such that

$$\bigcup_{j=1}^{n_2(m)} A_{2mj} \subseteq A_{2m}$$

and

$$A_{1m} \subseteq \bigcup_{j=1}^{n_1(m)} A_{1mj};$$

that the sets A_i, A_{im} and A_{imj} are all $\mathcal{H}^a$-measurable for $i = 1, 2$ each $a \in \mathbb{R}$ and appropriate $m, j \in \mathbb{N}$ and that there exist orthogonal transformations $\mathcal{T}_{m,j}^{A_1,A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ for $j = 1, ..., n_1(m)$ such that

$$\mathcal{T}_{mj}^{A_1,A_2}(A_{1mj}) \subseteq A_{2mj}.$$

If $A_1 \hookrightarrow^c A_2$ we write

$$C_n^{A_1,A_2} := \bigcup_{j=1}^{n_1(m)} \mathcal{T}_{m,j}^{A_1,A_2}(A_{1mj}).$$

Remark
For any $A_1, A_2 \in \mathcal{K}$ we can set $n_1(m) = n_2(m) = 2^m$ and for each $i \in \{1, 2\}$ $A_{im} := T_m := \cup_{j=1}^{2^m} T_{m,j}$ and $A_{imj} = T_{m,j}$. In this case, as we shall see, if $\theta_{n,i}^{A_1} \leq \theta_{n,i}^{A_2}$ for each n and i, we have, ignoring the negligible set of edge points E, $A_1 \hookrightarrow^c A_2$.

It would have been a simpler statement of definition to restrict to the case $A_1, A_2 \in \mathcal{K}$. However, as we shall see we will need to apply the definition where A_1 and A_2 are subsets of elements of $\mathcal{K}$ where certain triangular caps have been simply removed in the construction of A_1 and A_2. In any case, to make the definition intuitively easier to understand we may always think of each A_i as an element of $\mathcal{K}$ with triangular caps removed, each A_{im} as a union of a subcollection of the $T_{m,j}^{A_i}$ and each A_{imj} as a $T_{m,j}^{A_i}$.

In the case that A_1 and A_2 are actually in $\mathcal{K}$ we can restate the definition as follows

Definition 7.14. $\mathcal{K}$ version
Let A_1 and A_2 be A_ε type sets. We say that we can center A_1 in A_2 (or that A_1 can be centered in A_2) written $A_1 \hookrightarrow^c A_2$ if for each $n \in \mathbb{N}$ and $i \in \{1, .., 2^n\}$ there are orthogonal transformations $\mathcal{T}_{n,i}^{A_1,A_2}$ such that $\mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1}) \subset T_{n,i}^{A_2}$.

If $A_1 \hookrightarrow^c A_2$ then we write

$$C_n^{A_1,A_2} := \bigcup_{i=1}^{2^n} \mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1}).$$

Note that due to the fact that they are orthogonal transformations with both disjoint preimages and disjoint images we have both

$$\mathcal{H}_\delta^\eta(\mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1})) = \mathcal{H}_\delta^\eta(T_{n,i}^{A_1})$$

for each $i \in \{1, ..., 2^n\}$ and

$$\mathcal{H}_\delta^\eta\left(\bigcup_{i=1}^{2^n} \mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1})\right) = \mathcal{H}_\delta^\eta\left(\bigcup_{i=1}^{2^n} T_{n,i}^{A_1}\right)$$

for each $n \in \mathbb{N}$, for each pair $A_1 \hookrightarrow^c A_2$ and for each non negative $\eta, \delta \in \mathbb{R}$.

We now look at two properties of centering. The first is more a property of A_ε type sets that tells a condition allowing one A_ε type set to be centered into another. The second is a more general result showing that the dimension comparison works, thus justifying the use of centering.

Proposition 7.10. *Let A_1 and A_2 be A_ε type sets. Let $\theta_{n,\cdot}^{A_1}$ be denoted by $\theta_n^{A_1}$ and $\theta_{n,\cdot}^{A_2}$ be denoted by $\theta_n^{A_2}$ for each $n \in \mathbb{N}$. Then, if $T_{0,1}^{A_1} \subseteq T_{0,1}^{A_2}$ and $\theta_n^{A_1} \leq \theta_n^{A_2}$ for each $n \in \mathbb{N}$ then $A_1 \hookrightarrow^c A_2$.*

Proof:
We know that $T_{0,1}^{A_1} \subseteq T_{0,1}^{A_2}$ so that by denoting the identity transformation by ι we have $\mathcal{T}_{0,1}^{A_1,A_2} \equiv \iota$ and thus

$$\mathcal{T}_{0,1}^{A_1,A_2}(T_{0,1}^{A_1}) \subset T_{0,1}^{A_2}.$$

We then continue the proof by induction on n. Assume that

$$\mathcal{T}_{n,i}^{A_1,A_2}(T_{n,i}^{A_1}) \subset T_{n,i}^{A_2}$$

for some $n \in \mathbb{N}_0$ and each $i \in \{1, ..., 2^n\}$. Consider some arbitrarily chosen $j \in \{1, ..., 2^n\}$ with

$$\mathcal{T}_{n,j}^{A_1,A_2}(T_{n,j}^{A_1}) \subset T_{n,j}^{A_2}$$

and there fore since

$$diam(T_{n,i}^{A}) = \mathcal{H}^1(A_{n,i}^{A})$$

for each A_ε type set A it follows that

$$\mathcal{H}^1(A_{n,j}^{A_1}) \leq \mathcal{H}^1(A_{n,j}^{A_2}).$$

Now, $\theta_{n+1}^{A_1} \leq \theta_{n+1}^{A_2}$ by hypothesis and thus also, by Lemma 13

$$\begin{aligned}
\mathcal{H}^1(A_{n+1,2j+k}^{A_1}) &= \frac{1}{2}(cos(\theta_{n+1}^{A_1}))^{-1}\mathcal{H}^1(A_{n,j}^{A_1}) \\
&\leq \frac{1}{2}(cos(\theta_{n+1}^{A_2}))^{-1}\mathcal{H}^1(A_{n,j}^{A_1}) \\
&\leq \frac{1}{2}(cos(\theta_{n+1}^{A_2}))^{-1}\mathcal{H}^1(A_{n,j}^{A_2}) \\
&= \mathcal{H}^1(A_{n+1,2j+p}^{A_2})
\end{aligned}$$

for each $k, p \in \{-1, 0\}$.

Combining these, it follows that $T_{n+1,2j+k}^{A_1}$ can be mapped into $T_{n+1,2j+k}^{A_2}$ by placing $A_{n+1,2j+k}^{A_1}$ in the center of $A_{n+1,2j+k}^{A_2}$ for $k \in \{-1, 0\}$. By defining $\mathcal{T}_{n+1,2j+k}^{A_1,A_2}$ to be the orthogonal transformation that does this it follows that

$$\mathcal{T}_{n+1,2j+k}^{A_1,A_2}(T_{n+1,2j+k}^{A_1}) \subset T_{n+1,2j+k}^{A_2}$$

for $k \in \{-1, 0\}$. Since j was arbitrary we have $\mathcal{T}_{n+1,i}^{A_1,A_2}$ such that

$$\mathcal{T}_{n+1,i}^{A_1,A_2}(T_{n+1,i}^{A_1}) \subset T_{n+1,i}^{A_2}$$

for all $i \in \{1, ..., 2^{n+1}\}$, which completes the inductive step in n. $\diamondsuit$

We now prove the crucial step for the result we need to get our desired dimension results, saying that if one set can be centered in another then the expected result that it has a smaller dimension than the other holds.

Lemma 7.5.

$$A_1 \hookrightarrow^c A_2 \Rightarrow dim A_1 \leq dim A_2.$$

Proof:
Let $\eta > 0$ be such that $\mathcal{H}^\eta(A_2) = 0$.

Now, let $m \in \mathbb{N}$ then since $\mathcal{H}^\eta$ is invariant under orthogonal transformations we have

$$\begin{aligned}
\mathcal{H}^\eta(A_1) &= \mathcal{H}^\eta(A_1 \cap A_{1m}) \\
&= \mathcal{H}^\eta\left(\bigcup_{j=1}^{n_1(m)} A_1 \cap A_{1mj}\right) \\
&= \mathcal{H}^\eta\left(\bigcup_{j=1}^{n_1(m)} A_{1mj}\right) \\
&= \sum_{j=1}^{n_1(m)} \mathcal{H}^\eta(\mathcal{T}_{m,j}^{A_1,A_2}(A_{1mj})) \\
&\leq \sum_{j=1}^{n_1(m)} \mathcal{H}^\eta(A_{2mj}) \\
&= \mathcal{H}^\eta\left(\bigcup_{j=1}^{n_1(m)} A_{2mj}\right) \\
&\leq \mathcal{H}^\eta\left(\bigcup_{j=1}^{n_2(m)} A_{2mj}\right) \\
&\leq \mathcal{H}^\eta(A_{2m}).
\end{aligned}$$

We then find similarly for $m+1$

$$\mathcal{H}^\eta(A_1) \leq \mathcal{H}^\eta(A_{2(m+1)})$$

then since $A_{2(m+1)} \subset A_{2m}$ we have

$$\mathcal{H}^\eta(A_1) \leq \mathcal{H}^\eta(A_{2m} \cap A_{2(m+1)}),$$

by induction it follows that

$$\begin{aligned}
\mathcal{H}^\eta(A_1) &\le \mathcal{H}^\eta\left(\bigcap_{m=1}^{\infty} A_{2m}\right) \\
&\le \mathcal{H}^\eta(A_2) \\
&= 0.
\end{aligned}$$

As this is true for any $\eta \in \mathbb{R}$ for which $\mathcal{H}^\eta(A_2) = 0$ we have

$$\begin{aligned}
dim A_1 &= \inf\{\eta : \mathcal{H}^\eta(A_1) = 0\} \\
&\le \inf\{\eta : \mathcal{H}^\eta(A_2) = 0\} \\
&= dim A_2.
\end{aligned}$$

$\diamondsuit$

This completes the presentation of the necessary preliminary results and thus the chapter. In the following chapter we look at the theorems proving various results about the actual measure, rectifiability and dimension of A_ε type sets and Koch type sets.

Chapter 8

Dimension, Rectifiability and Measure of Generalised Koch type Sets

We now consider the main results for Koch type sets. That is under what conditions do we have finite, or weak locally finite measure. Under what conditions are Koch type sets rectifiable, or not rectifiable, and under what conditions can we determine the dimension of a set in $\mathcal{K}$. The results are all determined from the constrtuction parameters. All of the relevant parameters can be expressed in terms of the angles $\theta^A_{n,i}$. In the case of A_ε type sets we can exactly categorise the sets with respect to the above questions, for the Koch type sets it is not possible. The difference being that in the case of Koch type sets we could be generating measure from a pre-image set of measure zero in an otherwise well behaved set. The question of whether or not measure can indeed be generated remains at this time unanswered, the important point for us, is that it cannot be ruled out.

For this reason some of the results will continue to be stated seperately. In the general case we find, with respect to rectifiability, that,

$$A \in \mathcal{K} \text{ is countably 1-rectifiable } \Leftrightarrow \mathcal{H}^1(\{x : \tilde{\Pi}^A(x) = \infty\}) = 0.$$

With respect to measure, we find that for each $A \in \mathcal{K}$

$$\mathcal{H}^1(A) = \int_{A_0 \sim \Lambda_\infty^{-1}} \tilde{\Pi}^A d\mathcal{H}^1 + \mathcal{H}^1(\Lambda_\infty).$$

and that $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty) = \infty$. In general we would also expect $\mathcal{H}^1(\Lambda_\infty^{-1}) = 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty) = 0$ (that is the nongeneration of measure condi-

tion) so that we would then have

$$\mathcal{H}^1(A) = \int_{A_0} \tilde{\Pi}^A d\mathcal{H}^1.$$

While in certain cases (e.g. Λ_∞^{-1} is countable) it is certainly true, it may not be true in general. Note that this result holds also for $A \in \mathcal{K}$ with $dimA > 1$, in which case we get the uninformative result $\mathcal{H}^1(A) = \infty$.

Finally, with respect to dimension we define

$$\gamma_1^A := \sup\{a : \mathcal{H}^1(\{x : \tilde{\theta}_x^A \geq a\}) > 0\},$$

$$\gamma_2^A := \sup_{x \in A_0} \tilde{\theta}_x^A$$

and find

$$dim\Gamma_{f(\gamma_1^A)} = f_1(\gamma_1^A) \leq dimA \leq f_1(\gamma_2^A) = dim\Gamma_{f(\gamma_2^A)}$$

where

$$f(\gamma) := (1/2)(tan\gamma)$$

and therefore

$$f_1(\gamma) = -\frac{ln2}{ln((1/2)(1+(tan\gamma)^2)^{1/2})}.$$

Again, we find simplification under the hypothesis that for $B \subset A_0$ $\mathcal{H}^1(B) = 0 \Rightarrow \mathcal{H}^1(\mathcal{F}(B)) = 0$ in that we can then state

$$dimA \equiv f_1(\gamma_1^A).$$

It is in the A_ε type set case that we can ignore the possibilty of generalisation of measure and thus the "nicer" results can be stated for these sets.

8.1 Lipschitz Representation and Rectifiability

We start by showing that in some cases an A_ε type set is actually a Lipschitz graph, where $\mathcal{F}$ would pass as a Lipschitz function.

Lemma 8.1.
Suppose $A \in A^0$ and $\sum_{n=0}^\infty \theta_n^A < \infty$. Then for each $l > 0$ there is an $n_0 \in \mathbb{N}$ such that $A \cap T_{n_0,i}^A$ can be expressed as the graph of a Lipschitz function with Lipschitz constant less than or equal to l over $A_{n_0,i}^A$ for each $i \in \{1, ..., 2^{n_0}\}$.

Proof:
Let n_0 be such that

$$\sum_{n=n_0}^{\infty} \theta_n^A < \frac{tan^{-1}(l)}{5}.$$

Then let $x, y \in A \cap T_{n_0,i}^A$ for some $i \in \{1, ..., 2^{n_0}\}$ with $x \neq y$. We then know that there exists a $n_1 > n_0$ such that for each $n_0 \leq n < n_1$ $x, y \in T_{n,k}^A$ for some k and that $x \in T_{n_1,j}^A$ and $y \in T_{n_1,j\pm1}^A$ for some integer j. Without loss of generality let $x \in T_{n_1,j}^A$ and $y \in T_{n_1,j+1}^A$.

By choice of n_0 we know that

$$\psi_{A_{n_0,i}^A}^{A_{n_1,j}^A} < \frac{tan^{-1}(l)}{5}$$

and by Lemma 5.1

$$\psi_{T_{n_1,j+1}^A}^{T_{n_1,j}^A} < 2\theta_{n_1,j}^A < 2\frac{tan^{-1}(l)}{5}$$

so that when writing $X = \{z \in \mathbb{R}^2 : z = x + ty, t \in \mathbb{R}\}$

$$\psi_{A_{n_1,j}^A}^X < 2\psi_{T_{n_1,j+1}^A}^{T_{n_1,j}^A} < 4\frac{tan^{-1}(l)}{5}.$$

Thus

$$\psi_{A_{n_0,i}^A}^X < \frac{tan^{-1}(l)}{5} + 4\frac{tan^{-1}(l)}{5} = tan^{-1}(l)$$

and hence

$$\frac{|\pi_{(A_{n_0,i}^A)^\perp}(x) - \pi_{(A_{n_0,i}^A)^\perp}(y)|}{|\pi_{A_{n_0,i}^A}(x) - \pi_{A_{n_0,i}^A}(y)|} < tan(tan^{-1}(l)) = l.$$

Noting that (x, y) was an arbitrarily chosen pair of distinct points completes the proof. $\diamondsuit$

Combining this lipschitz result with Lemma 7.3 we are now able to present the rectifiability results. We first prove, both by Lipschitz graphs and the existence of approximate tangent spaces, the rectifiability under particular conditions of A_ε type sets. We present concurring with the philosophy that multiple proof methods allow further insight and understanding of the objects involved and are in any case interesting in their own right, as well as for comparative purposes.

We first prove the rectifiability using the Lipschitz lemmmas to show that certain A_ε type sets can then be expressed as $\mathcal{H}^1$-almost everywhere subsets of a countable union of Lipschitz graphs.

Theorem 8.1.
Whenever $A \in A^0$ satisfies $\sum_{n=0}^{\infty} \theta_n^A < \infty$, A is countably 1 rectifiable.

Proof:
Since $\sum_{n=0}^{\infty} \theta_n^A < \infty$ there is, by Lemma 8.1, an $n_0 \in \mathbb{N}$ such that for each $i \in \{1, ..., 2^{n_0}\}$ $A \cap T_{n_0,i}^A$ can be expressed as the graph of a Lipschitz graph over $A_{n_0,i}^A$. That is there is a Lipschitz function $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$A \cap T_{n_0,i}^A \subset f_i(A_{n_0,i}^A).$$

Then when $S_{n_0,i}^A : \mathbb{R} \to \mathbb{R}^2$ is a transformation satisfying

$$S_{n_0,i}^A([0, \mathcal{H}^1(A_{n_0,i}^A)]) = A_{n_0,i}^A$$

we can define $F_i : \mathbb{R} \to \mathbb{R}^2$ as $F_i = f_i \circ S_{n_o,i}^A$ to write

$$\begin{aligned} A &= \bigcup_{i=1}^{2^{n_0}} A \cap T_{n_0,i}^A \\ &\subseteq \bigcup_{i=1}^{2^{n_0}} f_i(A_{n_0,i}^A) \\ &\subset \bigcup_{i=1}^{2^{n_0}} F_i(\mathbb{R}). \end{aligned}$$

Since this is a subset of a form of expression of a set that is defined as being countably 1-rectifiable, the proof is complete. $\diamondsuit$

The second proof applies to sets with converging sums of base angles. In this case "potential" approximate tangent spaces eventually stop rotating and we can then use the approximate j-dimensionality to say that the set will be arbitrarily close to the limit of the rotating bases of the triangular caps containing a point and will thus have an approximate tangent space there.

Theorem 8.2.
Any $A \in A^0$ satisfying $\sum_{n=0}^{\infty} \theta_n^A < \infty$ has an approximate tangent space with multiplicity one almost eveywhere and is thus countably 1-rectifiable.

Proof:
We first prove that $A - E$ is countably 1-rectifiable. Let $y \in A - E$, write $H := \mathcal{H}^1(A)$ and let $f \in C_C^0(\mathbb{R}^2)$. It follows in particular that f is Lipschitz with Lipschitz constant F_1 and that there is an M such that

$$sptf \subset B_M(0).$$

Let $F = max\{1, F_1\}$. Since the other case is trivial we assume $M > 0$.

Let $\varepsilon > 0$ and define $\delta = \varepsilon/(MF)$. Since $A \in A^0$ we know that $A - E$ satisfies property (iv), we therefore know that there is a $\rho_y > 0$ such that for all $\rho \in (0, \rho_y]$ there is a $L_{y,\rho}$ such that

$$A \cap B_\rho(y) \subset L_{\rho,y}^{\delta\rho/2}$$

and we know in fact from the proof that $A - E$ satisfies (iv) that we may take $L_{y,\rho} || A^A_{n_\rho,i(y,n_\rho)}$ where $A^A_{n_\rho,i(y,n_\rho)}$ is taken such that $\mathcal{H}^1(A^A_{n_\rho,i(y,n_\rho)}) \in (\rho/2, \rho]$ and $y \in T^A_{n_\rho,i(y,n_\rho)}$.

Since $\sum_{n=0}^\infty \theta_n^A < \infty$ we know that $\{\psi_{\mathbb{R}}^{A^A_{n,i(n,y)}}\}$ is a convergent sequence and thus there is an affine space L such that

$$\psi_{\mathbb{R}}^L = \lim_{n\to\infty} \psi_{\mathbb{R}}^{A^A_{n,i(n,y)}}.$$

We then choose ρ_1 such that $\rho_1 < \rho_y$, so that for all $\rho < \rho_1$ the $A^A_{n_\rho,i(y,n_\rho)}$ taken as described above is such that

$$tan^{-1}(\psi^L_{A^A_{n_\rho,i(y,n_\rho)}}) < \frac{\delta}{2} \tag{8.1}$$

with n_ρ large enough for Lemma 8.1 to gaurantee that $A \cap T^A_{n_\rho,i(y,n_\rho)}$ can be expressed as the graph of a Lipschitz function with Lipschitz constant δ, and since $\sum_{n=0}^\infty \theta_n^A < \infty \Rightarrow \prod_{n=0}^\infty (cos\theta_n^A)^{-1} < \infty$ we take ρ_1 such that $n_{\rho_1}$is such that $\prod_{n=n_{\rho_1}}^\infty (cos\theta_n^A)^{-1} < 1 + \varepsilon$.

Now let $\lambda < \frac{\rho_1}{M}$. Then we have that $A \cap B_{\lambda M}(y) \subset (A^A_{n_\lambda,i(y,n_\lambda)})^{\delta\lambda M/2}$ so that by (8.1)

$$tan(\psi_L^{A^A_{n_\lambda,i(y,n_\lambda)}}) < \frac{\delta}{2}$$

so that

$$A \cap B_{M\lambda}(y) \subset L^{M\delta\lambda}$$

and thus

$$\eta_{y,\lambda} A \cap B_M(0) \subset L - y)^{M\delta}.$$

On this set we also have

$$|f(x) - f(\pi_L(x))| < Lipf \cdot \delta M \leq \frac{MF\varepsilon}{MF} = \varepsilon$$

for all $x \in \eta_{y,\lambda}(A - E)$.

By otherwise considering the positive and negative parts of f we may assume that $f \geq 0$. We then note

$$\int_{\eta_{y,\lambda}(A-E)} f(y) d\mathcal{H}^1(y) \leq \int_{\eta_{y,\lambda}(A-E)} \varepsilon d\mathcal{H}^1 + \int_{\eta_{y,\lambda}(A-E)} f(\pi_L(y)) d\mathcal{H}^1(y).$$

Then by Lemma 8.1 and Lemma 5.2 we know that we can apply the area formula with Jacobian calculated by taking the maximal vertical variation per unit along L as δ plus $2(2\theta^A_{n_\lambda})$. That is, with the Jacobian factor bounded above by $(1+9\delta^2)^{1/2}$ so that we have

$$\begin{aligned} \int_{\eta_{y,\lambda}(A-E)} f(y) d\mathcal{H}^1(y) &\leq \int_{\eta_{y,\lambda}(A-E)} \varepsilon d\mathcal{H}^1 + (1+9\delta^2)^{1/2} \int_L f(y) d\mathcal{H}^1(y) \\ &< \varepsilon \mathcal{H}^1(\eta_{n,\lambda}(A-E)) + (1+9\varepsilon) \int_L f(y) d\mathcal{H}^1(y) \end{aligned}$$

which implies

$$\begin{aligned} \left| \int_{\eta_{y,\lambda}(A-E)} f d\mathcal{H}^1 - \int_L f d\mathcal{H}^1 \right| &\leq \varepsilon(1+\varepsilon)2M + (1+9\varepsilon-1) \left| \int_L f d\mathcal{H}^1 \right| \\ &= \varepsilon(1+\varepsilon)2M + (9\varepsilon) \int_L f d\mathcal{H}^1. \end{aligned}$$

Since this is true for all $\varepsilon > 0$ it follows that

$$\lim_{\lambda \to 0} \left| \int_{\eta_{y,\lambda}(A-E)} f d\mathcal{H}^1 - \int_L f d\mathcal{H}^1 \right| = 0$$

so that

$$\lim_{\lambda \to 0} \int_{\eta_{y,\lambda}(A-E)} f d\mathcal{H}^1 = \int_L f d\mathcal{H}^1.$$

That is there is an approximate tangent space for y. Since this is true for all $y \in A - E$ and $\mathcal{H}^1(E) = 0$ we have

$$\begin{aligned} \lim_{\lambda \to 0} \int_{\eta_{y,\lambda} A} f d\mathcal{H}^1 &= \lim_{\lambda \to 0} \int_{\eta_{y,\lambda}(A-E)} f d\mathcal{H}^1 \\ &= \int_L f d\mathcal{H}^1 \end{aligned}$$

for all $y \in A - E$. That is, A has an approximate tangent space for all $y \in A - E$, and therefore $\mathcal{H}^1$-almost everywhere which implies that A is countably 1-rectifiable. $\diamondsuit$

Although these results are not for the entirety of A_ε type sets, the completion of the proofs of rectifiability falls under the proof for general $\mathcal{K}$ sets. We thus prove the more general result, stating the cleaner result for A_ε type sets as a Corollary.

Theorem 8.3.
Let $A \in \mathcal{K}$.
If $\mathcal{H}^1(\Lambda_\infty) = 0$ then A is countably 1-rectifiable.

Remark:
It would clearly be desireable to be able to show that

$$\mathcal{H}^1(\Lambda_\infty^{-1}) = 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty)$$

which would be an a better situation since we have better understanding, perception and control of sets in A_0 than sets in A. It is however not necessarily in general true (though it may be). We do in some limited cases have control from A_0. For example if Λ_∞^{-1} is countable then $\mathcal{H}^1(\Lambda_\infty) = 0$ and so the above Theorem would then state that with such a Λ_∞^{-1}, A is countably 1-rectifiable.

Proof:
We note that

$$\begin{aligned} A &= \Lambda_\infty \cup \bigcup_{m=1}^{\infty} \Lambda_m \\ &= \Lambda_\infty \cup \bigcup_{m=1}^{\infty} \mathcal{F}(\Lambda_m^{-1}) \\ &= \Lambda_\infty \cup \bigcup_{m=1}^{\infty} \mathcal{F}|_{\Lambda_m^{-1}}(\Lambda_m^{-1}). \end{aligned}$$

Since from Lemma 7.3 we know that $\mathcal{F}_{\Lambda_m^{-1}}$ is Lipschitz for each $m \in \mathbb{N}$ it follows that A is countably 1-rectifiable should $\mathcal{H}^1(\Lambda_\infty) = 0$. $\diamondsuit$

Before stating the corollary of rectifiability for A_ε sets, we prove the non-rectifiability result. In this way we will be able to demonstrate necessary and sufficient, that is, an equivalence of conditions for sets in A_ε to countable 1-rectifiability.

Theorem 8.4.
Let $A \in \mathcal{K}$ and $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0$. Then A is not countably 1-rectifiable.

Proof:
Let θ be any potential multiplicity function for A. Then $\theta \in L^1(\mathcal{H}^1, A, \mathbb{R})$ and thus θ is $\mathcal{H}^1$-measurable.

We then claim that there is an $r > 0$ such that

$$\mathcal{H}^1(\mathcal{F}^{-1}(\{x \in A : \theta(x) > r\}) \cap \Lambda_\infty^{-1}) > 0.$$

This is true for otherwise

$$\mathcal{H}^1(\{x \in A_{0,0} : \theta \circ \mathcal{F}(x) = 0\}) > 0$$

and thus

$$\mathcal{H}^1(\{x \in A : \theta(x) = 0\}) > 0$$

contradicting θ being a positive function on A. Set

$$B := \mathcal{F}^{-1}(\{x \in A : \theta(x) > r\}).$$

Since θ is measurable, $\{x \in A : \theta(x) > r\}$ is measurable and thus, since from Proposition 7.7 we know $\mathcal{F}$ is measurable, B is $\mathcal{H}^1$-measurable in $A_{0,0}$.

It then follows from Lemma 7.5 that there exists a $B_1 \subset B$ with $\mathcal{H}^1(B_1) = \mathcal{H}^1(B) > 0$ such that

$$\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), \mathcal{F}(x)) = \infty$$

for each $x \in B_1$.

Consider now $f \in C_C^0(\mathbb{R}^2, \mathbb{R})$ such that $\chi_{B_1(0)} \leq f \leq \chi_{B_2(0)}$ where χ is the characteristic function. Then for any tangent space, P, to A that may exist with respect to θ at $\mathcal{F}(x)$ for some $x \in B_1$

$$\theta(\mathcal{F}(x)) \int_P f(y) d\mathcal{H}^1(y) \leq \theta(\mathcal{F}(x)) \int_P \chi_{B_2(0)} d\mathcal{H}^1(y) = 2\theta(\mathcal{F}(x)) < \infty.$$

However

$$\begin{aligned}
\lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda} A} f(y)\theta(x+\lambda y) d\mathcal{H}^1(y) &\geq \lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda}\mathcal{F}(B_1)} f(y)\theta(x+\lambda y) d\mathcal{H}^1(y) \\
&\geq \lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda}\mathcal{F}(B_1)} \chi_{B_1(0)}\theta(x+\lambda y) d\mathcal{H}^1(y) \\
&> r \lim_{\lambda \searrow 0} \int_{\eta_{x,\lambda}\mathcal{F}(B_1)} \chi_{B_1(0)} d\mathcal{H}^1(y) \\
&\geq r\Theta^1(\mathcal{H}^1, \mathcal{F}(B_1), x) \\
&= \infty.
\end{aligned}$$

Thus

$$\lim_{\lambda\searrow 0}\int_{\eta_{x,\lambda}A} f(y)\theta(x+\lambda y)d\mathcal{H}^1(y)\neq\theta(\mathcal{F}(x))\int_P f(y)d\mathcal{H}^1(y).$$

Since this is true for any $x\in\mathcal{F}(B_1)$ and $\mathcal{H}^1(\mathcal{F}(B_1))\geq\mathcal{H}^1(B_1)>0$ it follows that A does not have an approximate tangent space with respect to θ at x on a set of x of positive measure.

Since this holds for any allowed selection of θ it follows from the definition of rectifiable sets and Theorem 3.1 that A is not countably 1-rectifiable. $\diamondsuit$

We can now state the cleaner result for A_ε type sets from which the particular results for A_ε and $\mathcal{A}_\varepsilon$ follow.

Corollary 8.1.
For an A_ε type set A, A is countably 1-rectifiable if and only if

$$\mathcal{H}^1((\Lambda_\infty^{-1})^A)=0.$$

Proof:
We note that A being A_ε type set implies $A\in\mathcal{K}$. Thus from Theorem 8.4, if $\mathcal{H}^1((\Lambda_\infty^{-1})^A)>0$ then A is not countably 1-rectifiable.

Conversely, Should $\mathcal{H}^1((\Lambda_\infty^{-1})^A)=0$ then there must exist at least one point, x, for which $\tilde{\Pi}_x^A\neq\infty$. Since $\tilde{\Pi}_x^A$ is constant for all $x\in A$ for an A_ε type set it follows that $\tilde{\Pi}_y^A\neq\infty$ for each $y\in A_{0,0}$ and thus for each $y\in A$. It follows that $\Lambda_\infty^A=\emptyset$ and therefore that $\mathcal{H}^1(\Lambda_\infty^A)=0$. It thus follows from Theorem 8.3 that A is countably 1-rectifiable. $\diamondsuit$

Theorem 8.5.
Let $\varepsilon>0$ and A be constructed as in Construction 3.2 with this ε. Then

$$\tilde{\Pi}_x^A\equiv\infty$$

and thus A is not 1-countably 1-rectifiable.

Proof:
From Lemma 7.2 we know that for any A_ε type set A_1,

$$\mathcal{H}^1(\tilde{A}_n{}^{A_1})=\mathcal{H}^1(A_{0,0}^{A_1})\prod_{j=0}^{n}(cos\theta_{j,\cdot}^{A_1})^{-1}=\prod_{j=0}^{n}(cos\theta_{j,\cdot}^{A_1})^{-1}.$$

Since from Lemma 3.2

$$\mathcal{H}^1(\tilde{A_n}^A) = (1 + n16\varepsilon^2)^{1/2}$$

it follows that

$$\begin{aligned}
\tilde{\Pi}^A &= \lim_{n\to\infty} \prod_{j=0}^{n} (cos\theta^A_{j,\cdot})^{-1} \\
&= \lim_{n\to\infty} \mathcal{H}^1(\tilde{A_n}^A) \\
&= \lim_{n\to\infty} (1 + n16\varepsilon^2)^{1/2} \\
&= \infty.
\end{aligned}$$

Thus $x \in (\Lambda_\infty^{-1})^A$ for each $x \in A_{0,0}$. This completes the first part of the proof.

It thus follows that $\mathcal{H}^1((\Lambda_\infty^{-1})^A) > 0$. From Proposition 7.7 (3) it then follows that $\mathcal{H}^1(\Lambda_\infty^A) > 0$. Therefore, from Corollary 8.1, A is not countably 1-rectifiable. $\diamondsuit$

The proof then that $\mathcal{A}_\varepsilon$ is not countably rectifiable that we present is an indirect proof, assuming that $\mathcal{A}_\varepsilon$ is countably 1-rectifiable, which then implies that A_ε is countably 1-rectifiable. This contradiction completes the proof and the rectifiability results.

Theorem 8.6.
For any appropriate $\varepsilon > 0$ for $\mathcal{A}_\varepsilon$ to be defined, $\mathcal{A}_\varepsilon$ is not countably 1-rectifiable.

Proof:
We prove the Theorem by contradiction. So, suppose that $\mathcal{A}_\varepsilon$ is countably 1-rectifiable and so can be written in the form

$$\mathcal{A}_\varepsilon \subset A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R})$$

where $\mathcal{H}^1(A_0) = 0$ and $F_n : \mathbb{R} \to \mathbb{R}^2$ is a Lipschitz function for each $n \in \mathbb{N}$.

We now consider that by the construction of A_ε we know that $A_\varepsilon \cap T_{i,j}$ is $A_{2^{1-i}\varepsilon}$ constructed on a base of length $\mathcal{H}^1(A_{i,\cdot})$ (which we note importantly is greater than 2^{1-i} so that should A_ε be well defined, then so too is the new A_ε).

It thus follows that by contradicting A_ε by 2^{1-i} in the vertical direction

and by $\mathcal{H}^1(A_{i,\cdot})$ in the horizontal direction we have that the result $C(A_\varepsilon)$ is a copy of any $A_\varepsilon \cap T_{i,j}$ (where C is the contraction map satisfying the said conditions).

We thus know that there exists contraction maps for each $i \in \mathbb{N}$ and $j \in \{1, ..., 2^i\}$, $O_{ij} : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$O_{ij}(A_\varepsilon) = A_\varepsilon \cap T_{i,j}$$

which implies

$$O_{ij}(\mathcal{A}_\varepsilon) \subset A_\varepsilon \cap T_{i,j}$$

and also that

$$O_{ij}(E) = E \cap T_{i,j}.$$

Define

$$M_{A_\varepsilon} := A_\varepsilon \cap \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{2^i} O_{ij}(\mathcal{A}_\varepsilon),$$

and

$$R_{A_\varepsilon} := A_\varepsilon \sim \left(\mathcal{A}_\varepsilon \cup \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{2^i} O_{ij}(A_\varepsilon) \right) = A_\varepsilon \sim M_{A_\varepsilon}.$$

It follows that

$$L_{ijn} := O_{ij}(F(x)) \qquad i, n \in \mathbb{N}, j \in \{1, ..., 2^i\}$$

are Lipschitz functions $L_{ijn} : \mathbb{R} \to \mathbb{R}^2$. We note that $\{\{L_{ijn}\}_{i,n\in\mathbb{N}}\}_{j=1}^{2^i}$ is countable. Also that R_{A_ε} ia a subset of the union of balls (or deformed balls) around points in E. Also that by taking the further addition to $\mathcal{A}_\varepsilon$, $O_{ij}(\mathcal{A}_\varepsilon)$, we infinitely reduce this area by continually refining the deformed ball around each e_n, that is

$$R_{A_\varepsilon} \subset \bigcup_{n\in\mathbb{N}} \bigcap_{\{i,j:O_{ij}((0,0))=e_n\}} O_{ij}(B_{r_1}((0,0))).$$

With this set up we can then attack the proof.

We first note that

$$M_{A_\varepsilon} \quad = \quad \mathcal{A}_\varepsilon \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^i} O_{ij}(\mathcal{A}_\varepsilon)$$

$$
\begin{aligned}
&\subset A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}\left(A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R})\right) \\
&= A_0 \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i}\bigcup_{n=1}^{\infty} L_{ijn}(\mathbb{R}) \\
&= A_0 \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0) \cup \bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i}\bigcup_{n=1}^{\infty} L_{ijn}(\mathbb{R}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}^1\left(A_0 \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0)\right) &\leq \mathcal{H}^1(A_0) + \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \mathcal{H}^1(O_{ij}(A_0)) \\
&\leq \mathcal{H}^1(A_0) + \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} \mathcal{H}^1(A_0) \\
&= 0 + \sum_{i=1}^{\infty}\sum_{j=1}^{2^i} 0 \\
&= 0.
\end{aligned}
$$

and $\bigcup_{n=1}^{\infty} F_n(\mathbb{R}) \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i}\bigcup_{n=1}^{\infty} L_{ijn}(\mathbb{R})$ is a countable collection of Lipschitz images.

It thus follows that M_{A_ε} is a countably 1-rectifiable. That is

$$M_{A_\varepsilon} = M_0 \cup \bigcup_{n=1}^{\infty} M_n(\mathbb{R})$$

where

$$M_0 = A_0 \cup \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{2^i} O_{ij}(A_0)$$

is a set of measure zero and $\{M_n\}_{n=1}^{\infty}$ is a reordering of $\{F_n\}_{n=1}^{\infty} \cup \{\{L_{ijn}\}_{i,n=1}^{\infty}\}_{j=1}^{2^i}$.

We now show that $\mathcal{H}^1(R_{A_\varepsilon}) = 0$.

Let $\eta > 0$. For each $i, n \in \mathbb{N}$ there exists $j_n = j_n(i,n) \in \{1, ..., 2^i\}$ such that $O_{ij}((0,0)) = e_n$. That is, $O_{ij}(B_{r_1}((0,0)))$ covers the part of R_{A_ε} centered on e_n, so that since

$$\lim_{i\to\infty} \mathcal{H}^1(A_{i,\cdot}) = 0$$

for each n we can choose an $i_n \in \mathbb{N}$ such that $diam(O_{ij}(B_{r_1}((0,0)))) < \eta 2^{-n}$. Then, since

$$R_{A_\varepsilon} \subset \bigcup_{n=1}^{\infty} O_{i_n j_n}(B_{r_1}((0,0)))$$

and since $diam(O_{ij}(B_{r_1}((0,0)))) < \eta 2^{-n} < \eta$ for each $n \in \mathbb{N}$ we then have that $\{O_{i_n j_n}(B_{r_1}((0,0)))\}_{n=1}^{\infty}$ is an appropriate covering set to estimate $\mathcal{H}^1_\eta$ and in fact we have

$$\begin{aligned}
\mathcal{H}^1_\eta(R_{A_\varepsilon}) &\leq \mathcal{H}^1_\eta\left(\bigcup_{n=1}^{\infty} O_{i_n j_n}(B_{r_1}((0,0)))\right) \\
&\leq \sum_{n=1}^{\infty} diam(O_{i_n j_n}(B_{r_1}((0,0)))) \\
&< \sum_{n=1}^{\infty} \eta 2^{-n} \\
&= \eta.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{H}^1(R_{A_\varepsilon}) &= \lim_{\eta \to 0} \mathcal{H}^1_\eta(R_{A_\varepsilon}) \\
&< \lim_{\eta \to 0} \eta \\
&= 0.
\end{aligned}$$

now since $\mathcal{A}_\varepsilon = M_{A_\varepsilon} \cup R_{A_\varepsilon}$ we have

$$\mathcal{A}_\varepsilon = R_{A_\varepsilon} \cup M_0 \cup \bigcup_{n=1}^{\infty} M_n(\mathbb{R}).$$

Since $\mathcal{H}^1(R_{A_\varepsilon}) = 0$,

$$\mathcal{H}^1(R_{A_\varepsilon} \cup M_0) = 0$$

and it follows that A_ε is countably 1-rectifiable. This contradicts Theorem 8.5, thus $\mathcal{A}_\varepsilon$ is not countably 1-rectifiable. $\diamondsuit$

This completes our study of rectifiability, we move on to the measure results before finally considering the dimension of Koch type sets.

8.2 Measure Formulae for Koch Type Sets

For our measure result we present, as previously seen, a formula that resembles the Area Formula. We could also have applied the Area Formula (for

more information on the Area Formula see for example Simon [15]) but not without some difficulty. We therefore present a self contained direct proof of the result.

Theorem 8.7.
Let $A \in \mathcal{K}$. Then, for all measurable $B \subset A_{0,0}$ the following holds

$$\mathcal{H}^1(\mathcal{F}(B)) = \int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 + \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty).$$

Remark:
As with the rectifiability theorem, the statement of this theorem would be simplified should it be true that

$$\mathcal{H}^1(\Lambda_\infty^{-1}) = 0 \Rightarrow \mathcal{H}^1(\Lambda_\infty) = 0$$

in which case we could write

$$\mathcal{H}^1(\mathcal{F}(B)) = \int_B \tilde{\Pi} d\mathcal{H}^1,$$

since, should $\mathcal{H}^1(\Lambda_\infty^{-1}) > 0$, both sides would then be ∞ so that they could in this case also be reconciled with one another.

It seems as though an application of the area formula for rectifiable sets is all that is necessary, which is likely to be true, however, since the convergence of $\tilde{\Pi}_n(x)$ is equivalent to the convergence of $\sum_n \theta_{n,i}^A(x)^2$ and thus not necessarily of $\sum_n \theta_{n,i}^A(x)$, the Jacobian is by no means a trivial quantity to calculate or show that it is equal to $\tilde{\Pi}$ on $A_{0,0} \sim \Lambda_\infty^{-1}$.

Proof:
We note that for any measurable $C \subset A_{0,0}$

$$\mathcal{F}(D) = \bigcap_{n=1}^{\infty} \bigcup_{i \in X_n} T_{n,i}^A$$

where

$$X_n := \{i \in \{1, ..., 2^n\} : i = i(n, x) \text{ for some } x \in D\}$$

and so can be constructed from countable unions and intersections of $\mathcal{H}^1$-measurable subsets of $\mathbb{R}^2$ and is therefore measurable. Also, since from Lemma 7.3 F_n is Lipschitz for each $n \in \mathbb{N}$ these sets are also measurable.

Further, since F_n is a Lipschitz map for each $n \in \mathbb{N}$, if $D \subset A_{0,0}$ so to is $F_n(D)$ for each $n \in \mathbb{N}$.

It follows then that

$$\mathcal{H}^1(\mathcal{F}(B)) = \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty) + \mathcal{H}^1(\mathcal{F}(B) \sim \Lambda_\infty).$$

We consider the second term.

Let $q \in \mathbb{N}$ and define

$$H_{nq} := \left\{ x \in A_0 : \frac{n-1}{q} < \tilde{\Pi}(x) \leq \frac{n}{q} \right\}.$$

We see

$$B \sim \Lambda_\infty^{-1} = \bigcup_{n=1}^{\infty} H_{nq}.$$

we now estimate $\mathcal{H}^1(\mathcal{F}(B \cap H_{nq}))$. Firstly $H_{nq} \subset \Lambda_{n/q}$ so that

$$\mathcal{F}(B \cap H_{nq}) = \mathcal{F}|_{\Lambda_{n/q}}(B \cap H_{nq})$$

is a Lipschitz graph with $\mathrm{Lip}\mathcal{F}|_{\Lambda_{n/q}} \leq n/q$ so that

$$\mathcal{H}^1(\mathcal{F}(B \cap H_{nq})) \leq \frac{n}{q}\mathcal{H}^1(H_{nq}).$$

It is now necessary to establish a lower estimate. To do this we define

$$H_{nqj} := \{x \in H_{nq} : \tilde{\Pi}_j(x) > (n-1)q^{-1} \geq \tilde{\Pi}_{j-1}(x)\}$$

and note that $H_{nqi} \cap H_{nqj} = \emptyset$ whenever $i \neq j$. We also define

$$J_{nq} := \{i \in \{1, ..., 2^j\} : i = i(n, x) \text{ for some } x \in H_{nqj}\}$$

We note that $\mathcal{F}|_{\Lambda_{n/q}} \circ F_j^{-1}$ is a Lipschitz expansion map on $F_j(\Lambda_{n/q})$. It follows that

$$\begin{aligned}
\mathcal{H}^1(\mathcal{F}(H_{nqj})) &= \mathcal{H}^1(\mathcal{F} \circ F_j^{-1} \circ F_j(H_{nqj})) \\
&= \mathcal{H}^1\left(\bigcup_{i \in J_{nq}} \mathcal{F}|_{\Lambda_{n/q}} \circ F_j^{-1}(F_j(H_{nqj}) \cap A_{j,i})\right) \\
&= \sum_{i \in J_{nq}} \mathcal{H}^1(\mathcal{F}|_{\Lambda_{n/q}} \circ F_j^{-1}(F_j(H_{nqj}) \cap A_{j,i}))
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i\in J_{nq}} \mathcal{H}^1(F_j(H_{nqj})\cap A_{j,i}) \\
&= \sum_{i\in J_{nq}} \tilde{\Pi}_{j,i}\mathcal{H}^1(H_{nqj}\cap[(i-2)2^{-j}, i2^{-j}]) \\
&> \sum_{i\in J_{nq}} \frac{n-1}{q}\mathcal{H}^1(H_{nqj}\cap[(i-1)2^{-j}, i2^{-j}]) \\
&= \frac{n-1}{q}\mathcal{H}^1(H_{nqj}).
\end{aligned}
$$

Since H_{nq} is the disjoint union of $\{H_{nqj}\}_{j=1}^{\infty}$ it follows that

$$
\begin{aligned}
\mathcal{H}^1(\mathcal{F}(H_{nq})) &= \sum_{j=1}^{\infty}\mathcal{H}^1(\mathcal{F}(H_{nqj})) \\
&> \frac{n-1}{q}\sum_{j=1}^{\infty}\mathcal{H}^1(H_{nqj}) \\
&= \frac{n-1}{q}\mathcal{H}^1(H_{nq}).
\end{aligned}
$$

It then follows that

$$\frac{n-1}{q}\mathcal{H}^1(H_{nq}) \leq \mathcal{H}^1(\mathcal{F}(H_{nq}) \leq \frac{n}{q}\mathcal{H}^1(H_{nq}).$$

Correspondingly we have direct from the definition of H_{nq} that

$$\frac{n-1}{q}\mathcal{H}^1(H_{nq}) < \int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 \leq \frac{n}{q}\mathcal{H}^1(H_{nq})$$

so that

$$\left|\int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(H_{nq})\right| < \frac{1}{q}\mathcal{H}^1(H_{nq})$$

and therefore

$$
\begin{aligned}
\left|\int_{B\sim\Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(B\sim\Lambda_\infty^{-1}))\right| &= \left|\sum_{n=1}^{\infty}\int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(H_{nq}))\right| \\
&\leq \sum_{n=1}^{\infty}\left|\int_{H_{nq}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(H_{nq}))\right| \\
&< \sum_{n=1}^{\infty}\frac{1}{q}\mathcal{H}^1(H_{nq})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{q}\mathcal{H}^1(B \sim \Lambda_\infty^{-1}) \\
&\leq \frac{1}{q}.
\end{aligned}
$$

Since this is true for all $q \in \mathbb{N}$ it follows that

$$\left| \int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 - \mathcal{H}^1(\mathcal{F}(B \sim \Lambda_\infty^{-1})) \right| = 0$$

and thus that

$$\int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 = \mathcal{H}^1(\mathcal{F}(B \sim \Lambda_\infty^{-1})).$$

This gives us

$$
\begin{aligned}
\mathcal{H}^1(\mathcal{F}(B)) &= \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty) + \mathcal{H}^1(\mathcal{F}(B) \sim \Lambda_\infty) \\
&= \int_{B \sim \Lambda_\infty^{-1}} \tilde{\Pi} d\mathcal{H}^1 + \mathcal{H}^1(\mathcal{F}(B) \cap \Lambda_\infty).
\end{aligned}
$$

$\diamondsuit$

As we mentioned at the beginning of this chapter, we present the simplified result for A_ε type sets. In this case, however, the result does not simplify. This is because, should $\tilde{\Pi}^A_\cdot \equiv \infty$ for some A_ε type set A then it could be this very A that allows for creation of measure. Then for any set $B \subset A_{0,0}$ with $\mathcal{H}^1(B) > 0$ we get

$$\int_B \tilde{\Pi} d\mathcal{H}^1 = \mathcal{H}^1(\mathcal{F}(B)) = \infty.$$

However, for a measurable set $B \subset A_{0,0}$ with $\mathcal{H}^1(B) = 0$ from which measure is created we would have

$$\int_B \tilde{\Pi} d\mathcal{H}^1 = 0$$

but

$$\mathcal{H}^1(\mathcal{F}(B)) > 0$$

preventing the simplified version of Theorem 8.7

$$\int_B \tilde{\Pi} d\mathcal{H}^1 = \mathcal{H}^1(\mathcal{F}(B))$$

holding as desired.

This, therefore, concludes our discussion of measure formulae and we now conclude with the results on dimension.

8.3 A Full Spectrum of Dimension

We complete this work with a discussion of the Dimension of A_ε and Koch type sets. As we discussed earlier in this Chapter, in order to gather results about dimension we essentially want to place sets either inside of or around sets that we know the dimension of. Unfortunately, generally with different A_ε type sets they do not generally stay neatly inside of one another. We therefore need to use our centralisation results to rearrange each stage of construction to ensure that strict containment is retained by the necessary sets.

As with the rectifiability results, the A_ε type sets allow for a more cleanly stated result than the Koch type sets. Unlike some of the previous result, we shall not prove the asthetically more pleasing results of the A_ε type sets as a corollary of the more general Koch type sets but shall rather prove the result directly. This is mainly because the proof attached to the A_ε type sets is much cleaner allowing the essential ingredients to be more clearly seen. The proof associated with the Koch type sets is then presented afterwards where the difficulties of allowing full variation of base angles require a much more technical proof.

As we will see from the results, a complete closed interval in $\mathbb{R}$ represents the possible dimensions of sets in $\mathcal{K}$. This shows the rich variation of the sets, which could otherwise perhaps have been of a dimension from a finite set of values.

Following the proof of the dimension of the A_ε type sets, we present a Corollary showing how the dimension of A_ε (which we directly proved to be 1 in Theorem 3.1) follows easily from the more general result.

Theorem 8.8. *For $r \geq 0$ and $A \in A^r$*

$$dimA = -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})}.$$

Proof:
The proof is dependent on the dimension of Γ_ε. We thererfore first note that for any scaling $\lambda \in \mathbb{R}$

$$dim\lambda\Gamma_\varepsilon = dim\Gamma_\varepsilon.$$

We also note that

$$\Gamma_{1/2(tanr)} \in A^r$$

and finally, recalling $dim\Gamma_\varepsilon = -ln2/(lnl)$ where l is the shrinking factor per approximation stage, we calculate that l_r, the approriate l for $\Gamma_\varepsilon \in A^r$ is

$$l_r = -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})}.$$

Now, since for $A \in A^r$ $\theta_n^A \searrow r$ we have $\theta_n^A \geq r$ for all $n \in \mathbb{N}$. Thus, since $\theta_n^{\Gamma_{1/2(tanr)}} \equiv r$ for all $n \in \mathbb{N}$ and thus also $\mathcal{H}^1(A_{0,1}^A)T_{0,1}^{\Gamma_{1/2(tanr)}} \subset T_{0,1}^A$ Proposition 7.10 then gives us that

$$\mathcal{H}^1(A_{0,1}^A)\Gamma_{1/2(tanr)} \hookrightarrow^c A.$$

Lemma 7.5 then gives

$$dimA \geq dim\mathcal{H}^1(A_{0,1}^A)\Gamma_{1/2(tanr)} = dim\Gamma_{1/2(tanr)}. \tag{8.2}$$

Then, for any $r_1 > r$ there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ $\theta_n^A \leq r_1$. It follows that by choosing arbitrarily and $j \in \{1, ..., 2^{n_0}\}$

$$\mathcal{H}^1(A_{n_0,j}^A)T_{0,1}^{\Gamma_{1/2(tanr_1)}} \supset T_{n_0,j}^A.$$

Now taking $T_j \in A^r$ to be the set generated by starting with $T_{n_0,j}^A$ and $\theta_n^{T_j} \equiv \theta_{n+n_0}^A$, we have by Proposition 7.10 that

$$T_j \hookrightarrow^c \mathcal{H}^1(A_{n_0,j}^A)\Gamma_{1/2(tanr_1)}.$$

It then follows from Lemma 7.5 that

$$dimT_j \leq dim\mathcal{H}^1(A_{n_0,j}^A)\Gamma_{1/2(tanr_1)} = dim\Gamma_{1/2(tanr_1)}.$$

Taking a finite union of such ses will not alter the dimension, thus

$$\begin{aligned} dimA &= dim\bigcup_{j=1}^{2^{n_0}} T_j \\ &= dimT_j \\ &\leq dim\Gamma_{1/2(tanr_1)} \\ &= -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})}. \end{aligned}$$

Since this is true for all $r_1 > r$ it follows that

$$dimA \leq -\frac{ln2}{ln(\frac{1}{2}(1+(tan(r))^2)^{1/2})} = dim\Gamma_{1/2(tanr)}.$$

Combining this with (8.2) gives the result $\diamondsuit$

Corollary 8.2.
$dimA_\varepsilon = 1$.

Proof:
Since from Proposition 7.5 we know $A_\varepsilon \in A^0$ for any given ε, we can directly apply Theorem 8.8 to calculate

$$\begin{aligned} dimA_\varepsilon &= -\frac{ln2}{ln(\frac{1}{2}(1+(tan(0))^2)^{1/2})} \\ &= -\frac{ln2}{ln(1/2)} \\ &= 1. \end{aligned}$$

$\diamondsuit$

Our final result is then the characterisation of dimension for the more general Koch type sets. As we see, the basic principle is the same as that used for A_ε type sets, the difference being the need to adjust for individually varying rates of change of base angle in the more general set up. We slowly eliminate those more rapidly decreasing, leaving those with a base measure enough to make a difference that reduce base angle slowly and would then, in the sense of Theorem 8.8 have higher dimension. It is these sets that dictate the dimension of the general whole set.

Theorem 8.9.
Let $A \in \mathcal{K}$ and

$$\gamma_1^A = \sup\{a : \mathcal{H}^1(\{x \in A_0 : \lim_{n\to\infty} \theta^A_{n,i(n,x)} \geq a\}) > 0\}$$

and

$$\gamma_2^A = \sup_{x\in A_0} \tilde{\theta}^A_x.$$

Then

$$dim\Gamma_{f(\gamma_1^A)} = f_1(\gamma_1^A) \leq dimA \leq f_1(\gamma_2^A) = dim\Gamma_{f(\gamma_2^A)}$$

where

$$f(\gamma) := (1/2)(tan\gamma)$$

and therefore

$$f_1(\gamma) := -\frac{ln2}{ln((1/2)(1+(tan\gamma)^2)^{1/2})}.$$

Should the hypothesis that for $B \subset A_0$ $\mathcal{H}^1(B) = 0 \Rightarrow \mathcal{H}^1(\mathcal{F}(B)) = 0$ hold, or should for a given $A \in \mathcal{K}$ we have $\mathcal{H}^1(\Upsilon_{\gamma_1^A+}) = 0$ then

$$dimA \equiv f_1(\gamma_1^A).$$

Proof:
We start by proving that $dimA \leq f_1(\gamma_1^A)$.

Let $\xi < \gamma_1^A$. Then $\mathcal{H}^1(\Upsilon_{\xi+}^{-1}) > 0$. There is therefore an $n_0 \in \mathbb{N}$ such that $\mathcal{H}^1(\Upsilon_{\xi+}^{-1}) > 2^{-n_0}$. It follows that

$$\Upsilon_{\xi+}^{-1} \cap [(i-1)2^{-n_0}, i2^{-n_0}] \neq \emptyset$$

for at least 2^{n-n_0} $i \in \{1, ..., 2^n\}$. It follows that

$$T_{n,i}^A \cap F_n(\Upsilon_{\xi+}^{-1}) \neq \emptyset$$

for at least 2^{n-n_0} $i \in \{1, ..., 2^n\}$.

In particular, this is true for all $n \geq n_0$. We set

$$A_{2m} := \cup\{T_{n_0+m,i}^A : T_{n_0+m,i}^A \cap F_{n_0+m}(\Upsilon_{\xi+}^{-1}) \neq \emptyset\}$$

and $n_2(m)$ to be the number of $T_{n_0+m,i}^A$ that are included in A_{2m}.

Note that $n_2(m) \geq 2^m$. Further we order these $T_{n_0+m,i}^A \subset A_{2m}$ as $\{A_{2mj}\}_{j=1}^{n_2(m)}$. We consider the set $\Gamma_{f(\xi)}$ constructed on a base $A_{0,0}$ of length

$$\mathcal{H}^1(A_0) = 2^{-n_0} \prod_{i=0}^{n_0-1} (cos\xi)^{-1}.$$

We denote this set by $\tilde{\Gamma}$.

We now want to show that

$$\tilde{\Gamma} \hookrightarrow^c A_2 := \bigcap_{m=1}^{\infty} A_{2m}.$$

Clearly, for any $T_{n_0+m+1,i}^A \subset A_{2(m+1)}$, $T_{n_0+m+1,i}^A \cap F_{n_0+m+1}(\Upsilon_{\xi+}^{-1}) \neq \emptyset$, also $T_{n_0+m+1}^A \subset T_{n_0+m,int(i/2)+1}^A$ so that

$$T_{n_0+m,int(i/2)+1}^A \cap F_{n_0+m+1}(\Upsilon_{\xi+}^{-1}) \neq \emptyset$$

and thus

$$T_{n_0+m,int(i/2)+1}^A \cap F_{n_0+m}(\Upsilon_{\xi+}^{-1}) \neq \emptyset,$$

so that $T_{n_0+m,\int(i/2)+1}^A \subset A_{2m}$ and hence we have $A_{2(m+1)} \subset A_{2m}$ for any $m \in \mathbb{N}$.

We see that in putting $\tilde{\Gamma}$ into the required form for Definition 7.13

$$\tilde{\Gamma} = A_1, T_n^{\tilde{\Gamma}} := \bigcup_{i=1}^{2^n} T_{n,i}^{\tilde{\Gamma}} = A_{1n}, T_{n,i}^{\tilde{\Gamma}} = A_{1ni}, \text{ and } n_1(m) = 2^m.$$

So that A_1 and A_2 individually satisfy the requirements of A_1 and A_2. Also, $n_1(m) = n_2(m)$. We therefore only need to show the existence of the transformations $\mathcal{T}_{n,i}^{A_1,A_2}$.

We note that each $A_{1mi} = T_{m,i}^{\tilde{\Gamma}}$ is a triangular cap of base length

$$2^{-m}\left(\prod_{i=0}^{m-1}(cos\xi)^{-1}\right) \times (\text{base length } T_{0,1}^{\tilde{\Gamma}})$$

which equals

$$2^{-m-n_0}\prod_{i=0}^{m+n_0-1}(cos\xi)^{-1}$$

and of base angle ξ.

We also note that for each $i \in \{1, ..., n_1(m)\}$, $i \in \{1, ..., n_2(m)\}$ so that A_{2mi} exists and is a triangular cap $T_{n_0+m,k}^{A}$ for some $k \in \{1, ..., 2^{n_0+m}\}$ with base angle $\theta_{n_0+m,k}^{A} \geq \xi$ and base length

$$2^{-n_0-m}\tilde{\Pi}_{n_0+m,k}^{A}.$$

Since a sequence $\{\theta_{n,i(n)}\}$ of angles in the construction of A is decreasing and $\theta_{n_0+m,k}^{A} \geq \xi$ it follows that

$$2^{-n_0-m}\tilde{\Pi}_{n_0+m,k}^{A} \geq 2^{-m-n_0}\prod_{i=0}^{m+n_0-1}(cos\xi)^{-1} = \mathcal{H}^1(A_{1mi}).$$

It follows, since A_{1mi} and A_{2mi} are isoceles triangles where A_{2mi} has a longer base and larger base angles that A_{2mi} is strictly larger than A_{1mi} in the sense that A_{1mi} could be placed inside of A_{2mi} and thus there must exist an orthogonal transformation $\mathcal{T}_{m,i}^{\tilde{\Gamma},A_2}$ such that

$$\mathcal{T}_{m,i}^{\tilde{\Gamma},A_2}(A_{1mi}) \subset A_{2mi}.$$

Since this is true for any $m \in \mathbb{N}$ and $i \in \{1, ..., n_1(m)\}$ it follows that $\tilde{\Gamma} \hookrightarrow^c A_2$.

Thus, using Lemma 7.5 and the fact that $A_2 \subset A$ we have

$$dim\Gamma_{f(\xi)} = dim\tilde{\Gamma} \leq dimA_2 \leq dimA.$$

Since this is true for each $\xi < \gamma_1^A$ it follows that

$$dimA \geq dim\Gamma_{f(\gamma_1^A)} = f_1(\gamma_1^A).$$

For the $\leq$ inequalities, we let $B \subset A_0$ be $\mathcal{H}^a$-measurable for each $a \in \mathbb{R}$ and show that for

$$\gamma = \sup_{x \in B} \tilde{\theta}_x^A$$

$$dim\mathcal{F}(B) \leq dim\Gamma_{f(\gamma)} = f_1(\gamma).$$

Let $\xi > \gamma$ and for each $n \in \mathbb{N}$ define

$$\chi_n := \cup\{T_{n,i}^A : \theta_{n,i}^A \geq \xi\}.$$

Then $\Psi_n := T_n - \chi_n$ is the finite union of triangular caps $T_{n,j}^A$ with $\theta_{n,j}^A \leq \xi$.

We see that for each such triangular cap $T_{n,j}^A \subset \Phi_n$,

$$\mathcal{H}^1(A_{n,j}^A) \leq \mathcal{H}^1(A_{0,1}^A) = \mathcal{H}^1(A_{0,1}^{\Gamma_{f(\xi)}})$$

and that for each later triangular cap $T_{n,m,k}^A \subset T_{n,j}^A$

$$\theta_{n+m,k}^A \leq \theta_{n,j}^A \leq \xi = \theta_{n+m,\cdot}^{\Gamma_{f(\xi)}}.$$

It therefore follows from Proposition 7.10 that for each $T_{n,j}^A \subset \Psi_n$

$$T_{n,j}^A \hookrightarrow^c \Gamma_{f(\xi)}$$

and hence, since $A \cap T_{n,j}^A$ equals the final set resulting from the Koch set construction starting fom $T_{n,j}^A$, Lemma 7.5 gives

$$dim(A \cap T_{n,j}^A) \leq dim\Gamma_{f(\xi)}$$

and therefore, since this is true for any such triangular cap, that

$$dim(A \cap \Psi_n) = dim(A \cap T_{n,j}^A) \leq dim\Gamma_{f(\xi)}.$$

Now, suppose that there exists a $y \in \mathcal{F}(B)$ with

$$y \notin \bigcup_{n=1}^{\infty} \Psi_n.$$

Then for each $n \in \mathbb{N}$ $\theta^A_{n,i(n,y)} \geq \xi$ and therefore

$$\tilde{\theta}^A_y = \lim_{n\to\infty} \theta^A_{n,i(n,y)} \geq \xi > \gamma.$$

Since this is impossible it follows that $\mathcal{F}(B) \subset \cup_{n=1}^{\infty}(\Psi_n \cap A)$ and therefore that

$$dim\mathcal{F}(B) \leq dim\Gamma_{f(\xi)}.$$

Since this is true for each $\xi > \gamma$ we have

$$dim\mathcal{F}(B) \leq dim\Gamma_{f(B)} = f_1(B).$$

To finish the proof we note that $f_1(\gamma) \geq 1$ for each $\gamma \geq 0$, and consider firstly that for each $x \in A_0$, $\tilde{\theta}^A_x \leq \gamma^A_2$ so that immediately from the above we have

$$dimaA \leq dim\Gamma_{f(\gamma^A_2)} = f_1(\gamma^A_2).$$

For the second conclusion we consider

$$B = \Upsilon^{-1}_{\gamma^A_1}.$$

It follows then that

$$dim\Upsilon_{\gamma^A_1+} \leq dim\Gamma_{f(\gamma^A_1)} = f_1(\gamma^A_1).$$

Should the hypothesis hold that for all $D \subset A_0$, $\mathcal{H}^1(D) = 0 \Rightarrow \mathcal{H}^1(\mathcal{F}(D)) = 0$, or should we directly have $\mathcal{H}^1(\Upsilon_{\gamma^A_1+}) = 0$, then we have $\mathcal{H}^1(\Upsilon_{\gamma^A_1+}) = 0$ and therefore

$$dim\Upsilon_{\gamma^A_1+} \leq dim\Gamma_{f(\gamma^A_1)} = f_1(\gamma^A_1).$$

We therefore have

$$\begin{array}{rcl} dimA & \leq & max\{dim\Upsilon_{\gamma^A_1}, \Upsilon_{\gamma^A_1+}\} \\ & \leq & dim\Gamma_{f(\gamma^A_1)} \\ & = & f_1(\gamma^A_1), \end{array}$$

which completes the proof $\diamondsuit$

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